IRE
Transactions

on INFORMATION THEORY

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IRE TRANSACTIONS®

on Information Theory

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Information Theory

Volume IT-3	June, 1957	Number 2

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F. Louis H. M. Stumpers

F. Louis H. M. Stumpers was born in Eindhoven, The Netherlands, on August 30, 1911. He joined the Philips Research Laboratories in 1928. From 1934 to 1937 he studied mathematics and physics at Utrecht University. After his doctoral examination and research work on Stark effect, he returned to the Philips Laboratories as a research physicist in 1938. His first work was concerned with the application of semiconductors in telecommunication. In 1939 he joined the Department of Fundamental Radio Research under Professor van der Pol. In this group he did work on circuit theory, cables, frequency modulation, stochastic problems, and noise. In 1946 he received the Doctoral degree in technical science from Delft University on a thesis about frequency modulation. Since then he did work on low-noise receivers, general noise problems and information theory, and radio-interference. In 1952–1953, on leave of absence from the Philips Laboratories, he was a research associate at the Research Laboratory of Electronics of the Massachusetts Institute of Technology.

From its conception in 1948, Dr. Stumpers has been a member of the board of the Foundation for the Study of Radioastronomy in the Netherlands.

For his part in the work that led to the registration of the hydrogen line in Kootwijk, he was awarded the Veder Radio Prize in 1951 (together with van der Hulst and Muller).

Since 1950 he has been a member of the Subcommittee for Information Theory of the International Scientific Radio Union (URSI). He is a member of the Netherlands URSI Committee.

Dr. Stumpers is vice-chairman of the Netherlands CISPR Committee, a member of International Subcommittee B (measurements), and a member of the Steering Committee of the international CISPR organization. (CISPR is the organization for the study and abatement of radio interference.)

He attended the CCIR Conference at Warsaw as an expert for The Netherlands delegation. Dr. Stumpers is the author of about twenty-five technical papers.

He is a member of the Institute of Radio Engineers and of the Administrative Committee of the Professional Group on Information Theory. He is also a member of the Dutch Physical Society, the Dutch Radio Society, and the study group for statistics of the Dutch language of the Dutch Society for Phonetics.

Information Theory and International Radio Organizations

F. LOUIS H. M. STUMPERS

In the expanding field of information theory time seems to run even faster than elsewhere in this fast moving world. In 1950, when the first London Symposium was held in the quiet rooms of the Royal Society, many of us came under the spell of a bright new idea, that might once again make possible a new synthesis, unifying many disciplines. Those present had the impression one could still know what was going on in the field of information theory, the men working in it, and its future possibilities. Only a few years have passed and information theory has had enormous successes, yet some feel disappointed. Once again specialization is taking its toll, making it impossible to follow more than a part of the field. Also, numerous as the papers are—and the yield of interesting and worthwhile papers is not less than in any other field—practical applications directly attributable to information theory are not yet great in number. This, together with a sound distrust of easy popularity has made for a more than critical attitude in some quarters.

Neither the disappointment nor the unusually critical attitude can, in my view, be justified. Information theory has stressed the use of some new tools and it has given us some new yardsticks. They can be applied in a great variety of situations. Only the specialist can assess their merit in his branch of science, *e.g.*, neurophysiology or optics, but we all gain by the understanding of mathematical models and methods and by the greater chance of cross-fertilization. Fields already well cultivated may not yield an extra crop, even so it is worth trying. The fact alone that we have better means to measure efficiency, does not of itself make the efficiency greater. That we had already fairly good qualitative means to compare systems, does not make a quantitative approach less worthwhile.

In the special field of radio communication many subjects were thoroughly studied before and quite apart from the application of communication theory. The International Radio Consultative Committee, CCIR, already had study groups in which complete radio systems as well as questions of bandwidth reduction, signal-to-noise ratio, fading, tolerable interference, and error detecting and correcting codes were studied. The steady progress in these subjects must have been favorably influenced by the theoretical developments, even if a direct link cannot always be shown. Though in CCIR emphasis is laid on practice rather than theory, it also issues at regular intervals a bibliography on communication theory with numerous abstracts. Communication theory is directly represented in the form of CCIR Question No. 133 and Study Program No. 86. The question concerns the investigation of technical methods for the most efficient use of communication channels. The study program suggests research on the comparison of existing codes with theoretical limits and the study of new codes, taking into account phenomena peculiar to radio propagation. As I can give here only an abridged version, those of our members interested should look at the original documents. Some very good studies in the literature provide at least a partial solution to these problems.

Another organization working on the application of science to radio is the International Scientific Radio Union. Since 1950 it has had a special commission for information theory under the chairmanship of Professor van der Pol, Director of CCIR until 1957. For the meeting in Boulder this summer the chairman has drawn attention to the problems of television-bandwidth reduction; error-detecting codes combined with automatic repeat request procedures; signal form, duration and bandwidth; and antennas for transhorizon communication.

A few of the many radiocommunication questions in which the ideas of information theory are fertile, have been mentioned above. Other subjects, *e.g.*, speech synthesis and visual perception are of obvious importance to this field.

Theoretical and fundamental studies come first, and much work remains to be done, but many will judge us by the practical applications we can find. Let us give special care to this. It is fair and right also to look outside the radio field, but in this Professional Group, radio still has the claim for first attention.

On the Detection of Stochastic Signals in Additive Normal Noise—Part I[*]

DAVID MIDDLETON†

Summary—The problem of optimum and suboptimum detection of normal signals in additive normal noise backgrounds is examined by the methods of statistical decision theory. Some general results for optimum receiver structure, error probabilities, and average risk are obtained for the case of colored noise backgrounds. A detailed study of threshold reception in white-noise backgrounds is included, along with calculations of Bayes risk, bias terms, and minimum detectable signals for broad-band RC-noise signals and narrow-band, *i.e.*, high-Q, LRC-noise signal processes. Optimum detector structures for signal processes with rational intensity spectra are also determined for the white noise case, and particular attention is paid to optimum receiver design in terms of physically realizable elements. Suboptimum receiver structure and performance are considered briefly, as well as a number of limiting cases of more special interest. General methods of attack are illustrated, with details given in Appendixes I-V. Application of the results to a variety of communication problems is indicated.

LIST OF SYMBOLS

a_0^2 = input signal-to-noise power ratio

$[B_m^{(T)}]_D$ = m^{th} iterated kernel for $D(t, T)$

$\mathbf{C}$ = optimum weighting for discrete sampling

C_α, C_β, etc. = preassigned constant costs

$\mathbf{D} = \mathbf{k}_S \mathbf{k}_N^{-1}$

$\mathfrak{D}_T(\gamma)$ = a Fredholm determinant

$F(\mathbf{V}|\mathbf{S})$ = conditional probability density of $\mathbf{V}$, given $\mathbf{S}$

$F_X(i\xi)_{N, S+N}, F_Y(i\xi)$ = characteristic functions

$\Theta(x)$ = error function: $\Theta(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$

$h(x)_R, h_T(x)$ = weighting functions of linear filters

$\mathbf{K}_N$ = noise covariance matrix; $\mathbf{k}_N$ = normalized covariance matrix

$\mathbf{K}_S$ = signal covariance matrix; $\mathbf{k}_S$ = normalized covariance matrix

$K(t, u)_{S, N}$ = covariance functions of signal and of noise

$\mathfrak{K}$ = a cost ratio, or threshold

L_n = n^{th} semi-invariant

$\log \mathfrak{L}_n(V)$ = a suboptimum system

$\mathbf{N}$ = noise vector

N_c, N_s, S_c, S_s = components of the noise and signal

$P_n(x), Q_n(x)$ = probability densities for $\log \Lambda_n(\mathbf{V})$

$P_T(x), Q_T(x)$ = error probabilities for $\log \Lambda_T$, continuous sampling

$R(\sigma, \delta)$ = average risk

$\mathfrak{R}_{1N}^*, \mathfrak{R}_{2N}^*$ = normalized Bayes risks

$\mathbf{S}$ = signal vector

T = duration of the observation period

V_c, V_s = components of the received data wave

$V_F(t)$ = output of a predetection filter

$\mathbf{V}$ = data vector

$\mathbf{v}$ = normalized data vector = $\mathbf{V}/\Psi_N^{1/2}$

W_{0N}, W_{0S} = spectral densities of the noise and signal processes, defined on the basis of a single-sided intensity spectrum and in terms of simple frequency (in cps)

$\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ = (column) vectors

$z_n^{(\alpha)}, z_n^{(\beta)}, z_T^{(\alpha)}, z_T^{(\beta)}, z_0', z_0''$ = arguments of the error probabilities

$z_T(t), \rho_T(t, u)$ = solutions of certain integral equations

α, β = conditional error probabilities

α^*, β^* = Bayes (conditional) error probabilities

α_M^*, β_M^* = Minimax conditional error probabilities

Γ_0, Γ_0', G_0 = bias terms

$\gamma_0^2 = W_{0S}/W_{0N}$

$\delta(\boldsymbol{\gamma} \mid \mathbf{V})$ = a decision rule

η = a numerical fraction $(0 \leq \eta \leq 1)$

$\Lambda_n(\mathbf{V})$ = generalized likelihood ratio, discrete sampling

$\Lambda_T(V)$ = generalized likelihood functional, continuous sampling

$\lambda = \omega_F T$ = normalized observation period

$\lambda_j^{(G)}$ = eigenvalues of the matrix $\mathbf{G}$

$[\lambda_j^{(T)}]_G$ = eigenvalues for the kernel G

$\mu = p/q$ = ratio of *a priori* probabilities of signal and of noise

σ_e^2 = effective input signal-to-noise (power) ratio

$\sigma_0^2 = \Psi_S T/W_{0N}$

Φ_n = structure factor for optimum system, discrete sampling

Φ_T = structure factor for optimum system, continuous sampling

Ψ_n = structure factor for suboptimum system, discrete sampling

Ψ_T = structure factor for suboptimum system, continuous sampling

Ψ_S, Ψ_N = mean intensity of signal and of noise

ω_F = parameter proportional to the spectral width of the signal process.

I. INTRODUCTION

ALTHOUGH deterministic signals—those whose general waveforms are known *a priori* at both transmitter and receiver—occur frequently in communication problems, signals with a wholly random character, while less common, are also of considerable practical interest. For example, random signals occur in the course of scatter-path transmission when originally

terministic signals are converted by the medium of propagation into (narrow-band) random waves whose velope and phase structure depend in part on the iginal signal. In the case of frequency-shift-key (fsk) mmunication, for instance, where the transmitted ave consists of sequences of sinusoidal wave-trains of ite duration at one or the other of two chosen carrier equencies, the original wave-trains are transformed to corresponding sequences of narrow-band, normal ise waves, centered about the original carrier frequency cations. Even if the medium of propagation does not ansform an original signal ensemble in such fashion, e signals themselves may possess such a complex time-ructure that a normal process is a reasonable approxi-ation to the actual signal set.

Other examples of random signal processes are easily und; waveforms of this type are to be expected in dio-spectroscopy, where now the source is represented y one or more spectral lines. Because of collision-broaden-g and other atomic effects, these lines have a finite idth, and the radiation process itself by which they re produced may be regarded as a normal process from e macroscopic point of view. A similar type of phe-omenon occurs in radioastronomy. There the signal urces are either radiostars or gas clouds whose signals re essentially normal noise distributed with varying tensities over the radio spectrum. The former produce oad-band noise, while the latter generate narrow-band ocesses. Still other examples of signals described by entirely random process can be readily constructed, ith counterparts in actual physical situations. From the ewpoint of communication theory, a task of central aportance in all these examples is the detection of the esence (or absence) of (one or more) such random gnals against an appropriate (normal) noise background.

Compared to the studies of optimum and suboptimum etection of deterministic signals[1] not much attention ppears to have been given to the corresponding problem random signals. Of previous work, the most significant this regard is that of Davis [2], who has given a rigorous nd rather general discussion of the problem but without otaining results in closed form or results that are readily asceptible of computation and interpretation as specific ceiver systems of realizable elements. Somewhat later, their analysis of the optimum sequential detection of gnals in noise, Bussgang and Middleton [3] considered iefly the question of Gaussian (broad-band) signals in milar, normal noise, but apart from the present discus-on, to the author's knowledge it is only in the closely lated work of Price [4] that an extensive treatment of e problem appears.[2]

Although this investigation and Price's interesting d important study have many points of contact and

profit mutually from their common origin in the under-lying problem of detecting normal noise in normal noise backgrounds, their approaches and emphases differ to a considerable extent. Price's work pursues the question of scatter-path communication to a much greater degree and considers more complicated conditions of operation, including fading, than does this study. Moreover, his treatment of narrow-band waves, based on the approach of Kac and Siegert [5], and of Emerson [6] in certain in-stances, is extended to include those more complicated situations and is considerably more thorough than the comparable analysis here. On the other hand, our treat-ment stems from the general approach of decision theory, as applied to detection, and in addition, includes an introductory account of the problem of colored noise in colored noise backgrounds, although no numerical results are presented at this time. We also consider the problem of broad-band noise signals, where attention is focused mainly on threshold reception. For these reasons, then, the two treatments supplement each other to a certain extent, with enough overlap to establish convenient analytical relations between the various different special problems examined in each study.

The purpose of this paper, accordingly, is to examine optimum and suboptimum systems for the detection of normal noise signals in additive, normal noise back-grounds by the methods of statistical decision theory [1]. Apart from special problems involving specific broad- and narrow-band signals and the systems appropriate to them, our aim is first, to outline a general approach for situations of this class in the usual cases of finite observation periods $(0, T)$ when the background noise is not "white," *i.e.*, does not possess a uniform spectrum at all frequencies; and second, to illustrate a new method of obtaining specific results in the important cases of threshold reception where the input signal energy is at most comparable to that of the interfering noise. Although this method is approximate, it is particularly well suited to the weak-signal cases and can be employed where an exact approach cannot be pressed further analytically.

Apart from these rather general aims, a number of new and more special results are obtained:

1) Solutions of the integral equations appropriate to signal ensembles with general rational spectra when the background noise is white;

2) Discrete structures for the general colored-noise case, as well as for white noise interference;

3) Minimax detection processes, when the *a priori* probability of signal and noise and of noise alone are unknown;

4) Specific, realizable receiver structures, in terms of linear (time-invariant) matched filters and zero-memory nonlinear elements, or time-varying linear elements and multiplier units;

5) A number of limiting conditions or operations, such as independent sampling (in the discrete cases) and infinite observation times.

Before we proceed to the main topics, let us first indicate

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[1] See Bibliography [1]. In particular, see references therein g. 250): (1.9), (1.10), (1.10a), (1.11), (1.12), (1.14), (1.15)-(1.20), .22)-(1.26), (1.30)-(1.34), (1.40), (1.48)-(1.50), (3.3), (3.12)-(3.17), .3).
[2] See also the related work of Turin [20].

the assumptions on which the analysis itself is based:

1) Both the noise background, $N(t)$, and the noise-signal, $S(t)$, belong to *independent normal* random processes, with zero means and with covariance functions $K_N(t_1, t_2)$, $K_S(t_1, t_2)$, respectively. In much of our later discussion we shall further assume that N and S are stationary processes, so that $K_N(t_1, t_2) = K_N(|t_1 - t_2|)$, etc., although this does not greatly reduce the generality of the treatment.

2) The background noise, $N(t)$, may be "white," with finite spectral density W_{0N}, or it may be "colored," with some nonuniform spectrum of bounded total intensity.

3) The noise-signal, $S(t)$, may be spectrally broad- or narrow-band; in any case, both signal and noise are additive processes.

4) Finite observation periods $(0, T)$ are postulated, except in the special situation of semi-infinite observation times $(T \to \infty)$.

Our basic problem, of course, is the binary one of determining whether or not the signal, $S(t)$, is present in the noise background. Statistically, this is equivalent to testing the hypothesis H_1 of $S + N$ against the alternative H_0 of N alone. Moreover, we shall take as our criterion of performance the measure of average risk or cost, R. For optimum performance, receiver structure and operation are determined by minimizing this average cost [1]. The quantities of chief interest to us, accordingly, are:

1) The structure of the receiver, which indicates how the received data is to be processed for a decision as to the presence or absence of a signal.

2) The "bias" term, which affects the decision threshold in actual operation.

3) The error probabilities, in terms of which average, or minimum average risk $(R, \text{ or } R^*)$, is calculated, and by which, in turn, performance of the system is evaluated. On the basis of R and R^* both optimum and suboptimum systems may be compared, with respect to a common criterion.[3]

Finally, we point out that there are important practical problems still remaining, to which the present effort is preliminary: solutions for colored noise backgrounds, the case where rms input signal level is unknown except for a distribution of possible values, the details of strong-signal operation, and certain useful suboptimum systems, all remain for a later study (to appear as Part II when completed).

II. General Structure of the Optimum Detector (Discrete Sampling)

A. Decision Theory Formulation

Following the earlier discussion of Middleton and Van Meter[4] we find that the formulation of our present problem of determining the presence or absence of a random signal, $S(t)$, in an (additive) noise background may be summarized as follows:

The optimum system is found by minimizing the average risk $R(\sigma, \delta)$, where

$$R(\sigma, \delta) = \int_\Omega d\mathbf{S}\sigma(\mathbf{S}) \int_\Gamma d\mathbf{V}F(\mathbf{V}|\mathbf{S}) \int_\Delta \delta(\boldsymbol{\gamma}|\mathbf{V})C(\mathbf{S}, \boldsymbol{\gamma})\, d\boldsymbol{\gamma}. \tag{1}$$

Here, specifically, we have

$\mathbf{V} = (V_1, \cdots, V_n)$, $V_k = V(t_k)$, a set of n sampled data values, arranged in order of increasing time, *i.e.*, $t_j > t_i$, if $j > i$. For convenience, sampling takes place at equal intervals in the observation period $(0, T)$, so that $t_k = kT/n$;

$\mathbf{S} = (S_1, \cdots, S_n)$, the sampled signal vector;

$F(\mathbf{V}|\mathbf{S}) = $ the conditional probability (density) for $\mathbf{V}$, given $\mathbf{S}$

$\sigma(\mathbf{S}) = $ the *a priori* probability density for $\mathbf{S}$. Here $\sigma(\mathbf{S}) = q\delta(\mathbf{S} - 0) + pw_1(\mathbf{S})$, where p, $q(= 1 - p)$, are the *a priori* probabilities of $S + N$ and N alone, respectively; $w_1(\mathbf{S})$ is the density function for $\mathbf{S}$, given the presence of S in noise N, where, of course, $\int w_1(\mathbf{S})d\mathbf{S} = 1$.

$\mathbf{N} = (N_1, \cdots, N_n)$—the sampled background noise vector.

$\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_m)$—a set of m decisions; here $\boldsymbol{\gamma} = (\gamma_0, \gamma_1)$, corresponding to $\gamma_0 \; \varepsilon \; N$ alone, $\gamma_1 \; \varepsilon \; S + N$. Thus, our decision problem is a *binary* one $(m = 2)$, as mentioned above.

$C(\mathbf{S}, \boldsymbol{\gamma}) = $ a set of costs, which are assigned to each possible combination of signal input to the receiver and decision output. In particular:

$$C(\mathbf{S} = 0; \gamma_0) = C_{1-\alpha}; \qquad C(\mathbf{S} \neq 0; \gamma_0) = C_\beta$$

$$C(\mathbf{S} = 0; \gamma_1) = C_\alpha; \qquad C(\mathbf{S} \neq 0; \gamma_1) = C_{1-\beta} \tag{2a}$$

for the four possible combinations of input states and final decisions. $C_{1-\alpha}$, $C_{1-\beta}$ are the costs associated with correct, or "successful" decisions, while C_α, C_β are the costs of incorrect, or "unsuccessful" decisions. From the definition of "successful" and "unsuccessful," it is clear that $C_\alpha > C_{1-\alpha}$, $C_\beta > C_{1-\beta}$. In addition, we set $C_{1-\alpha} \geq 0$, $C_{1-\beta} \geq 0$, although this is not a necessary restriction.

$\delta(\boldsymbol{\gamma}|\mathbf{V}) = $ a set of decision rules, by which decisions $\boldsymbol{\gamma}$ are made on the basis of the received data $\mathbf{V}$. In detection, δ is a *probability*, and for our problem we observe that there are but two situations $\delta(\gamma_0|\mathbf{V})$, $\delta(\gamma_1|\mathbf{V})$, subject to the condition that a definite decision is actually made, *viz*:

$$\delta(\gamma_0 \mid \mathbf{V}) + \delta(\gamma_1 \mid \mathbf{V}) = 1. \tag{2b}$$

The average risk may be expressed in terms of the *error probabilities* α, β, according to

$$R(\sigma, \delta) = \mathcal{R}_0 + q\alpha(C_\alpha - C_{1-\alpha}) + p\beta(C_\beta - C_{1-\beta}), \tag{3}$$

with $\mathcal{R}_0 \equiv qC_{1-\alpha} + pC_{1-\beta}$, and

[3] See Bibliography [1], in particular, (3.5), (3.6).
[4] See Bibliography [1], (3.1) and (3.2).

$$\equiv \int_\Gamma F(\mathbf{V} \mid 0) \, \delta(\gamma_1 \mid \mathbf{V}) \, d\mathbf{V};$$

$$\beta \equiv \int_\Gamma \langle F(\mathbf{V} \mid \mathbf{S}) \rangle_s \, \delta(\gamma_0 \mid \mathbf{V}) \, d\mathbf{V}. \qquad (4)$$

The α and β are the class conditional probabilities, respectively, of deciding a signal is present, or that noise alone occurs, when in either instance the reverse is actually true. Thus α and β are the probabilities of Type I and Type II errors, based on the condition of noise alone, and of signal and noise. The average $\langle \rangle_s$ in β is with respect to $w_1(\mathbf{S})$, governing $\mathbf{S}$, or if $\mathbf{S}$ is deterministic, any random parameters in $\mathbf{S}$.

Minimization of the average risk R_1, (3), follows from suitable choice of decision rule. It is found that[5] the optimum detector structure is the generalized likelihood ratio.

$$\Lambda_n(\mathbf{V}) = \mu \langle F(\mathbf{V} \mid \mathbf{S}) \rangle_s / F(\mathbf{V} \mid 0), \qquad \mu \equiv p/q, \qquad (5)$$

and the optimum decision process itself is

$$\left. \begin{array}{l} \text{decide } \gamma_1, \text{ i.e., } S + N, \text{ when } \Lambda_n \geq \mathfrak{K}, \\ \text{decide } \gamma_0, \text{ i.e., } N, \text{ when } \Lambda_n < \mathfrak{K} \end{array} \right\};$$

$$\mathfrak{K} \equiv \frac{C_\alpha - C_{1-\alpha}}{C_\beta - C_{1-\beta}} > 0. \qquad (6)$$

Here the cost ratio $\mathfrak{K}$ is called the *threshold* and depends only on the preassigned costs. (The subscript n on Λ_n reminds us that Λ_n, and $F = F_n$, also are functions of the sample-size or observation period.) Accordingly, the minimum average risk, or Bayes risk $R^* = \min_\delta R\,(\sigma, \delta)$, becomes [(3), (4)]

$$R^*(\sigma, \delta^*) = \mathfrak{R}_0 + q\alpha^*(C_\alpha - C_{1-\alpha}) + p\beta^*(C_\beta - C_{1-\beta}), \qquad (7)$$

where $\delta^*(\gamma_0 | \mathbf{V}) = 1$, if $\Lambda_n < \mathfrak{K}$, (6) and (2b). The (conditional) error probabilities for this minimum average risk system are obtained from (4), from the nature of the decision rule δ^*, above. Now, in application it is more convenient to deal with some monotonic function of Λ_n as the receiver operator on the data V. The most useful choice is the (natural) logarithm, so that henceforth here the optimum detector is, from (5),

$$\log \Lambda_n(\mathbf{V}) = \log \mu + \log \{\langle F(\mathbf{V} \mid \mathbf{S}) \rangle_s / F(\mathbf{V} \mid 0)\}. \qquad (8)$$

The error probabilities are now determined from

$$\alpha^* = \int_{\log \mathfrak{K}}^{\infty} Q_n(x) \, dx; \qquad \beta^* = \int_{-\infty}^{\log \mathfrak{K}} P_n(x) \, dx, \qquad (9)$$

where $Q_n(x)$, $P_n(x)$ are respectively the probability densities of $\log \Lambda_n(\mathbf{V})$ with respect to the hypotheses $H_0, H_1, i.e.,$[6]

$$Q_n(x) = \int_\Gamma \delta(x - \log \Lambda_n(\mathbf{V})) F(\mathbf{V} \mid 0) \, d\mathbf{V} \qquad (10a)$$

$$P_n(x) = \int_\Gamma \delta(x - \log \Lambda_n(\mathbf{V})) \langle F(\mathbf{V} \mid \mathbf{S}) \rangle_s \, d\mathbf{V}. \qquad (10b)$$

As has been shown previously[7] for particular choices of the cost ratio $\mathfrak{K}$ one gets either a Neyman-Pearson detection system, or Siegert's Ideal system (the latter for $\mathfrak{K} = 1$, corresponding to minimization of $\alpha q + \beta p$). Note, incidentally, that whenever we talk about *error* probabilities, we are implying a decision process of some sort, and consequently, some measure of value associated with the possible decisions, regardless of which evaluation function may be chosen (here embodied in the constant, preassigned costs).

B. *Colored Noise-Signal in Colored Noise Background*

For the additive, independent normal noises assumed in Section I, we have

$$F(\mathbf{V} \mid \mathbf{S}) = W_n(\mathbf{V} - \mathbf{S})_N = (2\pi)^{-n/2}(\det \mathbf{K}_N)^{-1/2}$$
$$\cdot \exp \{-\tfrac{1}{2}(\tilde{\mathbf{V}} - \tilde{\mathbf{S}}) \mathbf{K}_N^{-1} (\mathbf{V} - \mathbf{S})\}, \qquad (11)$$

with $F(\mathbf{V}|0) = W_n(\mathbf{V})_N$ obtained directly from (11) on setting $\mathbf{S} = 0$ therein; W_N is the distribution density of the noise alone. Also, we have

$$w_1(\mathbf{S}) = (2\pi)^{-n/2}(\det \mathbf{K}_S)^{-1/2} \exp \{-\tfrac{1}{2}\tilde{\mathbf{S}}\mathbf{K}_S^{-1}\mathbf{S}\} \qquad (12)$$

for our normal noise signals, where now (with $\overline{\mathbf{N}} = \overline{\mathbf{S}} = 0$)

$$\mathbf{K}_N = [K_N(t_j, t_k)] = \overline{[N(t_j)N(t_k)]};$$
$$\mathbf{K}_S = [K_S(t_j, t_k)] = \overline{[S(t_j)S(t_k)]} \qquad (13)$$

are the covariance matrices of the background noise and of the possible incoming signal. In both instances $\mathbf{K}_N$ and $\mathbf{K}_S$ are symmetric. By definition, it follows here that

$$\langle F(\mathbf{V} \mid \mathbf{S}) \rangle_s \equiv \int_s w_1(\mathbf{S}) W_n(\mathbf{V} - \mathbf{S})_N \, d\mathbf{S}, \qquad (14)$$

so that using (11) and (12), with the aid of [7]:

$$\int_{-\infty}^{\infty} \cdots \int \exp \{i\tilde{\mathbf{t}}\mathbf{x} - \tfrac{1}{2}\tilde{\mathbf{x}} \mathbf{A}\mathbf{x}\} \, d\mathbf{x}$$
$$= (2\pi)^{n/2}(\det \mathbf{A})^{-1/2} \exp \{-\tfrac{1}{2}\tilde{\mathbf{t}}\mathbf{A}^{-1}\mathbf{t}\} \qquad (15)$$

we get finally

$$\langle F(\mathbf{V} \mid \mathbf{S}) \rangle_s$$
$$= \frac{\exp \{-\tfrac{1}{2}\tilde{\mathbf{V}}\mathbf{K}_N^{-1}\mathbf{V} + \tfrac{1}{2}\tilde{\mathbf{V}}\mathbf{K}_N^{-1}(\mathbf{K}_S^{-1} + \mathbf{K}_N^{-1})^{-1}\mathbf{K}_N^{-1}\mathbf{V}\}}{(2\pi)^{n/2} \sqrt{\det \mathbf{K}_S \mathbf{K}_N(\mathbf{K}_S^{-1} + \mathbf{K}_N^{-1})}}. \qquad (16)$$

The optimum receiver structure for detection, (8), becomes

$$\log \Lambda_n = \log \mu - \tfrac{1}{2} \log \det (\mathbf{I} + \mathbf{K}_S\mathbf{K}_N^{-1})$$
$$+ \tfrac{1}{2}\tilde{\mathbf{V}}\{\mathbf{K}_N^{-1}(\mathbf{K}_S^{-1} + \mathbf{K}_N^{-1})^{-1}\mathbf{K}_N^{-1}\}\mathbf{V}. \qquad (17)$$

[5] See Bibliography [1], (3.2).
[6] For details, see Bibliography [1], (3.5).

[7] Bibliography [1], (3.3).

Note, incidentally, that since $\mathbf{N}$ and $\mathbf{S}$ are both Gaussian and independent, we may write alternatively

$$\langle F(\mathbf{V} \mid \mathbf{S})\rangle_s = W(\mathbf{V})_{S+N}$$

$$= \frac{\exp\{-\frac{1}{2}\tilde{\mathbf{V}}(\mathbf{K}_S + \mathbf{K}_N)^{-1}\mathbf{V}\}}{(2\pi)^{n/2}\sqrt{\det(\mathbf{K}_S + \mathbf{K}_N)}} \quad (18)$$

from which it follows that

$$\log \Lambda_n = \log \mu - \tfrac{1}{2}\log\det(\mathbf{I} + \mathbf{K}_S\mathbf{K}_N^{-1})$$

$$+ \tfrac{1}{2}\tilde{\mathbf{V}}(\mathbf{K}_N^{-1} - (\mathbf{K}_S + \mathbf{K}_N)^{-1})\mathbf{V}, \quad (19)$$

with the consequent identity

$$\mathbf{C} \equiv \mathbf{K}_N^{-1} - (\mathbf{K}_S + \mathbf{K}_N)^{-1} \equiv \mathbf{K}_N^{-1}(\mathbf{K}_S^{-1} + \mathbf{K}_N^{-1})^{-1}\mathbf{K}_N^{-1}. \quad (19a)$$

At this point we introduce the normalizations:

$$\mathbf{k}_S = \psi_S^{-1}\mathbf{K}_S; \quad \mathbf{k}_N = \psi_N^{-1}\mathbf{K}_N; \quad a_0^2 \equiv \psi_S/\psi_N, \quad (20)$$

where ψ_S and ψ_N are respectively the mean intensities, $\overline{S^2}$, $\overline{N^2}$, of signal and noise, while a_0^2 is the *input signal-to-noise power ratio*. The optimum detector structure, (17), then becomes, for discrete sampling,

$$\log \Lambda_n = \log \mu - \tfrac{1}{2}\log\det(\mathbf{I} + a_0^2\mathbf{D})$$

$$+ \left(\frac{\psi_N}{2}\right)\tilde{\mathbf{v}}\mathbf{C}\mathbf{v}, \quad \mathbf{D} \equiv \mathbf{k}_S\mathbf{k}_N^{-1}, \quad (21)$$

where $\mathbf{v}$ is the normalized data $\mathbf{V}/\Psi_N^{1/2}$ and $\mathbf{C}$ is given by (19a). The terms $\Gamma_0 \equiv \log\mu - \tfrac{1}{2}\log\det(\mathbf{I} + a_0^2\mathbf{D})$ (196) are called the *bias*, while

$$\Phi_n \equiv \psi_N\tilde{\mathbf{v}}\mathbf{C}\mathbf{v} = \tilde{\mathbf{V}}\mathbf{C}\mathbf{V}, \quad (22)$$

is termed the *structure* of the optimum detector. Thus, more compactly, our optimum system is

$$\log \Lambda_n = \Gamma_0 + \tfrac{1}{2}\Phi_n = \Gamma_0(a_0^2; n) + \tfrac{1}{2}\Phi_n(a_0^2, \mathbf{v}). \quad (23)$$

C. Weak- and Strong-Signal Cases

Observe first of all that since reception is necessarily incoherent here[8] and because of the normal statistics of signal and noise, the detector structure involves at most a generalized autocorrelation of the received data $\mathbf{V}$ with itself, through $\Phi_n = \psi_N\tilde{\mathbf{v}}\mathbf{C}\mathbf{v}$. Not only for weak signals ($a_0^2 \stackrel{<}{\sim} 1$), but for strong ones as well ($a_1^2 \gg 1$) is this true: the optimum receiver is essentially an energy detector. Let us, however, for the moment consider the weak-signal case. For this it is convenient to use the first form of $\mathbf{C} = \mathbf{C}(a_0^2)$, (19a), which on expansion yields for the structure factor

$$\Phi_n(a_0^2; \mathbf{v}) = \tilde{\mathbf{v}}\left\{\sum_{l=1}^{\infty}(-1)^{l+1}a_0^{2l}\mathbf{D}^l\mathbf{k}_N^{-1}\right\}\mathbf{v}$$

$$\doteq a_0^2\tilde{\mathbf{v}}\{\mathbf{D}\mathbf{k}_N^{-1}\}\mathbf{v} + 0(a_0^4), \quad a_0^4 \ll 1, \quad (24)$$

revealing the expected dependence on the input signal-to-noise *power* ratio a_0^2 (instead of a_0, for coherent systems).

The bias term, Γ_0, may be similarly developed. From the results of Appendix III, (197) we have

$$\Gamma_0(a_0^2) = \log\mu + \sum_{m=1}^{\infty}\frac{(-1)^m}{2m}a_0^{2m}\text{ trace }\mathbf{D}^m$$

$$\doteq \log\mu - (a_0^2/2)\text{ trace }\mathbf{D} + 0(a_0^4) \quad (25)$$

and the series is convergent, provided $a_0^2 \mid \lambda_1^{(D)} \mid < 1$, where $\lambda_1^{(D)}$ is the largest eigenvalue[9] of the matrix $\mathbf{D} = \mathbf{k}_s\mathbf{k}_N^{-1}$. [The condition for the convergence of (24) is that $\Phi_n(a_0^2)$ be convergent, with respect to either hypothesis H_0 or H_1, see (10a), (10b)].

In the strong-signal situation we may use the second form of the identity (19a) for $\mathbf{C}(a_0^2)$ to obtain the following expression for the optimum detector's structure:

$$\Phi_n(a_0^2; \mathbf{v}) \simeq \tilde{\mathbf{v}}\mathbf{k}_N^{-1}\mathbf{v}$$

$$+ \tilde{\mathbf{v}}\sum_{l=1}^{\infty}a_0^{-2n}(-1)^n\mathbf{k}_N^{-1}(\tilde{\mathbf{D}}^{-1})^n\mathbf{v}, \quad a_0^2 \gg 1, \quad (26)$$

which shows that to a first approximation, the structure Φ_n is independent of a_0^2. This, as well as the dependence of Φ_n on a_0^2 in the weak-signal cases, is an example of the phenomenon of *modulation suppression* whereby a weak component in the presence of a strong is made still weaker in the course of rectification. Here the noise is suppressed[10] ($\mathbf{V} \simeq \mathbf{S}$) while in the threshold situation it is the signal that is suppressed.[11] The bias term, on the other hand, becomes to a first approximation

$$\Gamma_0 \cong \log\mu - \tfrac{1}{2}\log\det(a_0^2\mathbf{D}), \quad a_0^2 \gg 1. \quad (27)$$

For a development in inverse powers of a_0^2, (26), some procedure other than (197) for the weak-signal case must be used.

III. Error Probabilities and Bias (Discrete Sampling)

A. Eigenvalue Method

Before considering specific conditions of operation, let us find the error probabilities, and hence the minimum average risk, associated with our optimum detection process. We consider first discrete sampling. From (9), (10), and (7) we may obtain the Bayes risk, with the help of (23) for the particular problem of the present paper. We may write the following relations for the characteristic function of $Q_n(x)$, $P_n(x)$—the distribution densities of $\log\Lambda_n$, now considered as a random variable—when $\mathbf{V}$ is allowed ensemble properties:

$$\int_{-\infty}^{\infty} Q_n(x)e^{i\xi x}\,dx = F_x(i\xi)_N;$$

$$\int_{-\infty}^{\infty} P_n(x)e^{i\xi x}\,dx = F_x(i\xi)_{S+N} \quad (28)$$

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[8] Bibliography [1], (3.8).

[9] In all our work it is assumed that the eigenvalues of the noise, signal, bias, and other pertinent matrices of the theory, are distinct.

[10] The noise, of course, still influences the *structure* of the receiver, through $\mathbf{k}_n^{-1}$; similar remarks apply for the signal in the threshold cases.

[11] Bibliography [1], (3.8).

om (10), and (23), or (27), and the fact that

$$(x - \log \Lambda_n(\mathbf{V}))$$

$$= \int_{-\infty}^{\infty} \exp \left\{ -i\xi[x - \Gamma_0 - \tfrac{1}{2}\Phi_n(a_0, \mathbf{v})] \right\} \frac{d\xi}{2\pi}. \qquad (29)$$

hus' we have

$$F_x(i\xi)_N = \langle \exp \left\{ +i\xi(\Gamma_0 + [\psi_N/2]\bar{\mathbf{v}}\mathbf{C}\mathbf{v}) \right\} \rangle_{H_0};$$

$$F_x(i\xi)_{S+N} = \langle \exp \left\{ i\xi(\Gamma_0 + [\psi_N/2]\bar{\mathbf{v}}\mathbf{C}\mathbf{v}) \right\} \rangle_{H_1}, \qquad (30)$$

here the average with respect to the hypothesis H_0 is ade with the weighting function $F(\mathbf{V} \mid 0)$, and for H_1 ith the weighting function $\langle F(\mathbf{V} \mid \mathbf{S}) \rangle_s$. Applying (11), $= 0$, and (16) or (18) $[\mathbf{S} \neq 0]$, with the help of (15), we et directly

$$F_x(i\xi)_N = e^{i\xi\Gamma_0}[\det (\mathbf{I} - i\xi\mathbf{k}_N\psi_N\mathbf{C})]^{-1/2}, \qquad (31a)$$

$$F_x(i\xi)_{S+N} = e^{i\xi\Gamma_0}[\det (\mathbf{I} - i\xi\psi_N(a_0^2\mathbf{k}_S + \mathbf{k}_N)\mathbf{C})]^{-1/2}, \qquad (31b)$$

hile the corresponding probability densities are

$$P_n(x) = \int_{-\infty}^{\infty} \frac{e^{i\xi(\Gamma_0 - x)}}{\{\det (\mathbf{I} - i\xi\psi_N\mathbf{k}_N\mathbf{C})\}^{1/2}} \frac{d\xi}{2\pi} \qquad (32a)$$

$$P_n(x) = \int_{-\infty}^{\infty} \frac{e^{i\xi(\Gamma_0 - x)}}{\{\det (\mathbf{I} - i\xi\psi_N(a_0^2\mathbf{k}_S + \mathbf{k}_N)\mathbf{C})\}^{1/2}} \frac{d\xi}{2\pi}. \qquad (32b)$$

he error probabilities themselves follow from (9) after nother quadrature. At this point it is convenient to rite [see (19a)]

$$\psi_N\mathbf{k}_N\mathbf{C} = \mathbf{I} - (\mathbf{I} + a_0^2\mathbf{D})^{-1} \equiv \mathbf{G}_N;$$

$$\psi_N(a_0^2\mathbf{k}_S + \mathbf{k}_N)\mathbf{C} = a_0^2\mathbf{D} \equiv \mathbf{G}_{S+N}. \qquad (33)$$

There are two ways of simplifying the determinental xpressions in (31) and (32). The first method is that mployed originally by Kac and Siegert [5], which, in ffect, diagonalizes $\mathbf{G}$ and replaces the determinant by s eigenvalue product, (146). Thus, we can write for the eterminants in (32a) and (32b) above

$$\det (\mathbf{I} - i\xi\mathbf{G}) = \prod_{j=i}^{n} (1 - i\xi\lambda_j^{(G)}). \qquad (34)$$

owever, even with this reduction, the expressions for and P_n above cannot be evaluated precisely. At best, e can use the method of steepest descents to obtain an pproximation result which, fortunately, is useful in reshold cases. For details, see Appendix II.

For the binary detection problems considered here it also possible to obtain $P_n(x)$ directly from $Q_n(x)$, and ce versa, with the help of the fact that [21]

$$\cdot \left[\frac{\langle F(\mathbf{V}|\mathbf{S}) \rangle_s}{F(\mathbf{V}|0)} \right]^m F(\mathbf{V}|0) \, d\mathbf{V}$$

$$= \int_{\Gamma} \left[\frac{\langle F(\mathbf{V}|\mathbf{S}) \rangle_s}{F(\mathbf{V}|0)} \right]^{m-1} \langle F(\mathbf{V}|\mathbf{S}) \rangle_s \, d\mathbf{V} \qquad (34a)$$

here $m = 1, 2, \cdots$. In the logarithmic case here, *e.g.*, $= \log \Lambda_n$. Starting with (32a), for example, and using 3), straightforward manipulation yields

$$Q_n(x) = [\det (\mathbf{I} + a_0^2\mathbf{D})]^{1/2}$$

$$\cdot \int_{-\infty}^{\infty} \frac{e^{i\xi(\Gamma_0 - x)}}{(\det [\mathbf{I} - (i\xi - 1)a_0^2\mathbf{D}])^{1/2}} \frac{d\xi}{2\pi}, \qquad (34b)$$

which with an obvious substitution becomes finally[12]

$$Q_n(x) = [\det (\mathbf{I} + a_0^2\mathbf{D})]^{1/2} e^{\Gamma_0 - x}$$

$$\cdot \int_{-\infty}^{\infty} \frac{e^{i\xi'(\Gamma_0 - x)}}{\{\det (\mathbf{I} - i\xi'a_0^2\mathbf{D})\}^{1/2}} \frac{d\xi'}{2\pi}$$

$$= \mu e^{-x} P_n(x). \qquad (34c)$$

B. Trace Method

The second method, which to the author's knowledge has not been exploited in problems of this type before, is the so-called *trace method* (See Appendix II), which makes use of the basic identity

$$\det (\mathbf{I} + \gamma\mathbf{G}) = \exp \left\{ -\sum_{m=1}^{\infty} \frac{(-1)^m}{m} \gamma^m \text{ trace } \mathbf{G}^m \right\}, \qquad (35)$$

where the series converges, provided $| \gamma\lambda_1^{(G)} | < 1$, with $\lambda_1^{(G)}$ the largest eigenvalue of $\mathbf{G}$. This approach is distinguished by the fact that it does not require detailed knowledge of the eigenvalues of $\mathbf{G}$, only of the various traces of $\mathbf{G}$, $\mathbf{G}^2$, etc. Here $\gamma = -i\xi$, and both members of (35) are to be raised to the $-1/2$ power in (32).

When a_0^2 is small, we may replace the determinant by the trace expansion (35) above, approximatley for all ξ in our expressions for the probability densities P_n, Q_n of (32). Retaining terms in ξ, ξ^2 only in the exponents we develop the integrand in a series in $(i\xi)$, which upon integration with the help of (188) gives us finally

$$Q_n(x) \simeq \frac{e^{-z^2 N/2}}{\sqrt{2\pi(L_2)_N}} + 0\left(\frac{L_3}{L_2^{3/2}}\right)_N, \qquad (36a)$$

$$P_n(x) \simeq \frac{e^{-z^2 S+N/2}}{\{2\pi(L_2)_{S+N}\}^{1/2}} + 0(L_3/L_2^{3/2})_{S+N}. \qquad (36b)$$

Here L_2, L_3 are the second and third semi-invariants [see (180) and (186a)] and

$$\begin{cases} z_N = \dfrac{x - \Gamma_0 - \tfrac{1}{2} \text{ trace } \mathbf{G}_N}{(\text{trace } \mathbf{G}_N^2/2)^{1/2}} \, ; \\[2ex] (L_2)_N = \tfrac{1}{2} \text{ trace } \mathbf{G}_N^2 \end{cases}$$

$$\begin{cases} z_{S+N} = \dfrac{x - \Gamma_0 - (\tfrac{1}{2}) \text{ trace } \mathbf{G}_{S+N}}{(\tfrac{1}{2} \text{ trace } \mathbf{G}_{S+N}^2)^{1/2}} \, ; \quad \text{and} \\[2ex] (L_2)_{S+N} = \tfrac{1}{2} \text{ trace } \mathbf{G}_{S+N}^2. \end{cases} \qquad (37)$$

As expected, the distribution of the logarithm of the detector structure, including bias, (23), is normal, with mean $\Gamma_0 + (1/2)$ trace $\mathbf{G}$ and variance $(1/2)$ trace $\mathbf{G}^2$. Correction terms, showing that the series is essentially of the Edgeworth type, may be found from (189) and (189a) with (186a) and the appropriate expressions, (33) for $\mathbf{G}$.

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[12] The author is indebted to Dr. Price for calling this to his attention.

The Bayes error probabilities in the threshold case accordingly follow from (37) in (9), and are

$$\alpha^* \simeq \tfrac{1}{2}[1 - \Theta(z_n^{'(\alpha)}/\sqrt{2})];$$

$$\beta^* \simeq \tfrac{1}{2}[1 + \Theta(z_n^{(\beta)}/\sqrt{2})], \qquad (38)$$

when $\Theta(u) = (2/\sqrt{\pi}) \int_0^u e^{-t^2} dt$ is the familiar error integral, and

$$z_n^{(\alpha)} \equiv \frac{\log \mathcal{K} - \Gamma_0 - (\tfrac{1}{2})\,\text{trace } \mathbf{G}_N}{\{\text{trace } \mathbf{G}_N^2 /2\}^{1/2}} \,;$$

$$z_n^{(\beta)} \equiv \frac{\log \mathcal{K} - \Gamma_0 - (\tfrac{1}{2})\,\text{trace } \mathbf{G}_{S+N}}{\{\text{trace } \mathbf{G}_{S+N}^2 /2\}^{1/2}}. \qquad (39)$$

Correction terms are easily found from (189) on integration. Note that by an expansion of the type

$$\{\det(\mathbf{I} - i\xi\mathbf{G})\}^{-1/2} = \prod_{j-1}^{n} (1 - i\xi\lambda_j^G)^{-1/2}$$

$$\doteq e^{i\xi A_1 - \xi^2 A_2/2}\{1 + 0(i\xi)^3\} \qquad (40)$$

and using the relation $\sum_{j=1}^n \lambda_j^m = \text{trace } \mathbf{G}^m$, see (148), we can also obtain the weak-signal results above, (36)-(39). As we shall see in Section VII, there is an alternative approach when the signal process is narrow-band that enables us to use the Kac-Siegert method of resolution into eigenvalues and so obtain exact results for all a_0^2. For details, see Appendix I.

There remains the bias term, Γ_0, for this situation of discrete sampling. The exact expression is given by

$$\Gamma_0 = \log \mu - (1/2) \log \det (\mathbf{I} + a_0^2 \mathbf{D}),$$

$$\mathbf{D} = \mathbf{k}_S \mathbf{k}_N^{-1}, \quad \text{or} \qquad (41a)$$

$$= \log \mu + \prod_{j=1}^{n} [1 + a_0^2 \lambda_j^{(D)}]^{-1/2}, \qquad (41b)$$

[(21) *et seq.*] and for the weak-signal cases we may use (197) once more, to write

$$\Gamma_0 = \log \mu - \frac{a_0^2}{2} \text{trace } \mathbf{D}$$

$$+ \frac{a_0^4}{4} \text{trace } \mathbf{D}^2 + 0(a_0^6), \qquad (a_0^6 \ll 1). \qquad (41c)$$

As will become evident when Γ_0 is used in the expressions above for $z_n^{(\alpha)}$, $z_n^{(\beta)}$, it is necessary to consider terms $0(a_0^4)$ as well. Thus, for threshold reception, discrete sampling of colored noise signals received in colored noise, we get with the help of (192a) in (39)

$$z_n^{(\alpha)} = \frac{\log (\mathcal{K}/\mu) + \sum_{q=2}^{\infty} (-1)^q \left(\dfrac{q-1}{2q}\right) a_0^{2q} \text{trace } \mathbf{D}^q}{\left(\sum_{q=1}^{\infty} \dfrac{(-1)^q}{2} b_{2,q} a_0^{2q} \text{trace } \mathbf{D}^q\right)^{1/2}},$$

$$(a_0^2 < 1) \qquad (42a)$$

and from (193).

$$z_n^{(\beta)} = \frac{\log (\mathcal{K}/\mu) + \sum_{q=2}^{\infty} \dfrac{(-1)^{q+1}}{2q} a_0^{2q} \text{trace } \mathbf{D}^q}{a_0^2 [(\text{trace } \mathbf{D}^2/2)]^{1/2}},$$

$$(a_0^2 \stackrel{<}{\sim} 1). \qquad (42b)$$

The Bayes error probabilities are then obtained when (42) is used in (38). Estimates of the useful ranges of α^*, β^*, (38), may be established with the assistance of (189), (189a), where the various semi-invariants may be evaluated from Appendix III-A, especially (192a), (192b), and (193).

IV. Optimum Reception with Colored Noise Backgrounds (Continuous Sampling)

A. Calculation of Optimum Detector Structure and Bias

While we shall not pursue in any detail the question of colored noise backgrounds in the present paper, leaving this for a possible later study (Part II), it may be instructive to indicate the initial steps for this more general theory when continuous, rather than discrete sampling procedures are employed. We begin with the structure term $\Phi_n = \tilde{\mathbf{V}}\mathbf{C}\mathbf{V}$, (22), and by certain matrix manipulations, followed by an appropriate passage to the limit $n \to \infty$, with T fixed, we obtain the continuous versions of the discrete cases outlined in Section III.

Here let us use the second relation for $\mathbf{C}$ in (19a) writing

$$\mathbf{X} = \mathbf{K}_N^{-1}\mathbf{V}, \qquad (43)$$

we see that the structure factor Φ_n becomes

$$\Phi_n = \tilde{\mathbf{X}}(\mathbf{K}_S^{-1} + \mathbf{K}_N^{-1})^{-1}\mathbf{X}$$

$$= \tilde{\mathbf{X}}\mathbf{G}^{-1}\mathbf{X}, \quad \text{with} \quad \mathbf{G} \equiv \mathbf{K}_S^{-1} + \mathbf{K}_N^{-1}. \qquad (44)$$

Now we set

$$\mathbf{Y} = \mathbf{G}^{-1}\mathbf{X}, \quad \text{and} \quad \therefore \mathbf{X} = \mathbf{G}\mathbf{Y} = (\mathbf{K}_S^{-1} + \mathbf{K}_N^{-1})\mathbf{Y}. \qquad (45)$$

Our next step is to define a new column vector $\mathbf{Z}$ by

$$\mathbf{Z} \equiv \psi_N \mathbf{K}_N^{-1}\mathbf{Y}, \qquad (46)$$

from which we see with the help of (43) and (44) in $\mathbf{Y} = \mathbf{G}^{-1}\mathbf{X}$ that, alternatively,

$$\mathbf{Z} = \psi_N \mathbf{C}\mathbf{V}. \qquad (47)$$

Thus, we can write the structure factor in several equivalent ways:

$$\Phi_n = \tilde{\mathbf{X}}\mathbf{Y} = \psi_N^{-1}\tilde{\mathbf{X}}\mathbf{K}_N\mathbf{Z} = \psi_N^{-1}\tilde{\mathbf{V}}\mathbf{Z}. \qquad (48)$$

The matrix equations from which the integral equations are derived in the continuous case are given by (43) and (45) in conjunction with (46). We have

$$\mathbf{V} = \mathbf{K}_N\mathbf{X}; \qquad \mathbf{K}_S\mathbf{X} = \psi_N^{-1}(\mathbf{K}_N + \mathbf{K}_S)\mathbf{Z}. \qquad (49)$$

At this point we assume that $V(t)$ and a finite number of its derivatives are bounded and continuous functions of t and that $Z(t)$ is also a bounded, continuous function such that dZ/dt exists in the interval $(0 -, T +)$, except for possible δ-function singularities at $t = 0$, $t = T$. Then by letting the maximum interval $(t_j < t < t_j + \delta)$ approach zero as $n \to \infty$, $(\delta \to 0)$, with $\lim_{n\to\infty}$, $\delta \to 0$ $(j\delta = t_j) \to$

or all t in $(0-, T+)$, we note formally that

$$\lim_{\substack{n\to\infty \\ \delta\to 0}} \left\{ \frac{X(t_i + \delta) - X(t_i)}{\delta} \right\} = \frac{dX(t)}{dt} \equiv x(t), \qquad (50a)$$

$$\lim_{\substack{n\to\infty \\ \delta\to 0}} \left\{ \frac{Z(t_i + \delta) - Z(t_i)}{\delta} \right\} = \frac{dZ(t)}{dt} \equiv \zeta(t) \qquad (50b)$$

hen (49) goes over into the pair of integral equations

$$V(t) = \int_{0-}^{T+} K_n(t, u) \cdot dX(u) = \int_{0-}^{T+} K_n(t, u) x(u)\, du, \qquad (51a)$$

$$\int_{0-}^{T+} K_s(t, u) x(u)\, du = \psi_N^{-1} \int_{0-}^{T+} [K_S(t, u) + K_N(t, u)] \zeta(u)\, du \qquad (51b)$$

$$(0- < t < T+).$$

olution of the first relation, (51a), may be carried out y the methods described in Appendix IV-B, (214) nd (215), when the signal and noise processes possess ational spectra. Repeating this approach for the second elation, (51b), gives us $\zeta(t)$ as well. Now from (48) we ave for the structure factor under these conditions

$$\text{m } \Phi_n \to \Phi_T = \psi_N^{-1} \int_{0-}^{T+} V(t)\, dZ(t)$$

$$= \psi_N^{-1} \int_{0-}^{T+} V(t) \zeta(t)\, dt, \quad \text{or} \qquad (52a)$$

$$= \psi_N^{-1} \iint_{0-}^{T+} x(t) K_n(t, u) \zeta(u)\, dt\, du. \qquad (52b)$$

Ve emphasize that (50)-(52) are a set of purely formal perations, which must be justified in particular cases. Iowever, the kernels K_S, K_N encountered in our present hysical problems appear such as to permit definite and nique results on passage from the discrete to the con- nuous sampling processes, as evidenced by solution in articular instances. (See Appendix IV for white noise ackgrounds.)[13]

The bias (41) now becomes for continuous sampling rom the argument presented in Appendix I-A),

$$_0 = \log \mu - \tfrac{1}{2} \log \mathfrak{D}_T(a_0^2)$$

$$= \log \mu + \log \prod_{j=1}^{\infty} (1 + a_0^2 [\lambda_j^{(T)}]_D)^{-1/2}, \qquad (53)$$

here $\mathfrak{D}_T(a_0^2)$ is the Fredholm determinant $\lim_{n\to\infty} \det$ $+ a_0^2 \mathbf{D})$, $\mathbf{D} = \mathbf{k}_S \mathbf{k}_N^{-1}$. Here $[\lambda_j^{(T)}]_D$ are the eigenvalues f $\mathbf{D}$ in the limit $(n \to \infty)$, see Appendix I-B, (147) seq. In threshold reception, provided $a_0^2 |[\lambda_1^{(T)}]_D|^{-1} < 1$, Appendix II, (161a)-(161c)] we may alternatively repre- ent Γ_0 in terms of the iterated kernels,

$$B_m^{(T)})]_D$$

$$= \int_0^T \cdots \int D(t_1, t_2) D(t_2, t_3) \cdots D(t_m, t_1)\, dt_1 \cdots dt_m, \qquad (54)$$

ith the aid of the basic identity (161b), which is here pecifically, for $\gamma = a_0^2$,

$$\Gamma_0 = \log \mu + \sum_{m=1}^{\infty} \frac{(-1)^m}{2m} a_0^{2m} [B_m^{(T)}]_D;$$

$$a_0^2 |[\lambda_1^{(T)}]_D|^{-1} < 1. \qquad (55)$$

See Appendix III-B and (198).

We can now indicate the complete structure of the optimum detector when continuous sampling techniques are employed. Instead of the function $\log \Lambda_n(\mathbf{V})$ of $\mathbf{V}$,

we have instead the functional of $V(t)$,

$$\lim_{n\to\infty} \log \Lambda_n \to \log \Lambda_T(V(t)) = \log \mu - \tfrac{1}{2} \log \mathfrak{D}_T(a_0^2)$$

$$+ \tfrac{1}{2} \Phi_T(V(t); a_0^2), \quad \text{all } a_0^2 \geq 0, \qquad (56)$$

from (53) and (52). In the weak-signal situation we can use (55) alternatively.

In a similar way, instead of $Q_n(x)$ and $P_n(x)$ as the distribution densities of $\log \Lambda_n$, (10a), (10b) and (32a), (32b), we have for the corresponding distribution densities of the functional $\log \Lambda_T$, the relations

$$Q_T(x) = \int_{-\infty}^{\infty} \frac{e^{i\xi(\Gamma_0 - x)}}{\mathfrak{D}_T(-i\xi)_N^{1/2}} \frac{d\xi}{2\pi}$$

$$= \int_{-\infty}^{\infty} e^{i\xi(\Gamma_0 - x)} \prod_{j=1}^{\infty} (1 - i\xi [\lambda_j^{(T)}]_N)^{-1/2} \qquad (57a)$$

$$P_T(x) = \int_{-\infty}^{\infty} \frac{e^{i\xi(\Gamma_0 - x)}}{\mathfrak{D}_T(-i\xi)_{S+N}^{1/2}} \frac{d\xi}{2\pi}$$

$$= \int_{-\infty}^{\infty} e^{i\xi(\Gamma_0 - x)} \prod_{j=1}^{\infty} (1 - i\xi [\lambda_j^{(T)}]_{S+N})^{-1/2}. \qquad (57b)$$

The first expressions in (57a), (57b) are the analogs of (47a), (47b) for discrete sampling. Here $[\lambda_j^{(T)}]_N$ and $[\lambda_j^{(T)}]_{S+N}$ are the eigenvalues of the integral equations

$$\int_0^T G(t, \tau)_{N, S+N} f_i(\tau)_T\, d\tau = [\lambda_i^{(T)}]_{N, S+N} f_i(t)_T,$$

$$(0 \leq t \leq T) \qquad (58)$$

where $\mathbf{G}_N$, $\mathbf{G}_{S+N}$, (33), become the kernels $G(t, \tau)_{N, S+N}$ here; see Appendix I-B and (152a) *et seq.*

B. *Error Probabilities; Threshold Reception*

The Bayes error probabilities α^*, β^* follow at once (as a pair of quadratures), if we insert (57a), (57b) into (9), in place of P_n, Q_n. However, the same difficulties are encountered here as in the discrete situation, since it is not possible to evaluate $Q_T(x)$, $P_T(x)$ in general. The threshold cases can be treated, on the other hand, with the aid of the basic identity (161b), *viz.*

$$\mathfrak{D}_T(-i\xi) = \exp \left\{ \sum_{m=1}^{\infty} \frac{(-1)}{m} (i\xi)^m [B_m^{(T)}] G \right\},$$

$$\xi |[\lambda_1^{(T)}]_G|^{-1} < 1, \qquad (59)$$

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[13] For a discussion of this point, see Bibliography [1], (3.8).

where $[B_m^T]_G$ is the mth iterated kernel of $\mathbf{G}_N$ or $\mathbf{G}_{S+N}$, (33), as these matrices go over into $G(t, \tau)_{N, S+N}$ for continuous sampling; $[\lambda_1^{(T)}]_G$ is the largest eigenvalue of (58), (149a). Replacing the Fredholm determinant by (59) and retaining terms $0(\xi, \xi^2)$ only in the exponent, we may evaluate (57a), (57b) approximately, to obtain the analogs of (36) in the discrete case, *viz*:

$$Q_T(x) \simeq \exp\left\{-(z_N^{(T)})^2/2\right\}\left\{2\pi(L_2)_N\right\}^{-1/2};$$
$$P_T(x) \simeq \exp\left\{-z_{S+N}^{(T)2}/2\right\}\left\{2\pi(L_2)_{S+N}\right\}^{-1/2}, \qquad (60)$$

where now, in place of (37) for $z^{(T)}$, we have specifically

$$z_N^{(T)} = \left\{x - \Gamma_0 - \tfrac{1}{2}[B_1^{(T)}]_N\right\}(L_2)_N^{-1/2};$$
$$z_{S+N}^{(T)} = \left\{x - \Gamma_0 - \tfrac{1}{2}[B_1^{(T)}]_{S+N}\right\}(L_2)_{S+N}^{-1/2} \qquad (61a)$$

with

$$(L_2)_N = \tfrac{1}{2}[B_2^{(T)}]_N; \qquad (L_2)_{S+N} = \tfrac{1}{2}[B_2^{(T)}]_{S+N}. \qquad (61b)$$

Here $[B_1^{(T)}]_N$ is the first iterated kernel for $G(t, \tau)_N$, etc. Correction terms to the normal distribution densities may be found as before with the aid of (188), (189), Appendix II-C. A similar development of the eigenvalue forms in (57a), (57b) using (149a) also results in (60) and (61), as the reader can readily verify for himself.

In the threshold situation, we get approximately for the minimum error probabilities

$$\alpha^* \simeq \tfrac{1}{2}\{1 - \Theta(z_T^{(\alpha)}/\sqrt{2})\};$$
$$\beta^* \simeq \tfrac{1}{2}\{1 + \Theta(z_T^{(\beta)}/\sqrt{2})\}, \qquad (62)$$

and with the aid of (55) and (59) we find that now

$$z_T^{(\alpha)} = \left[\log\left(\mathcal{K}/\mu\right) - \frac{1}{2}\left\{[B_1^{(T)}]_N\right.\right.$$
$$\left.\left. + \sum_{m=1}^{\infty}\frac{(-1)^m}{m}a_0^{2m}[B_m^{(T)}]_D\right\}\right]\left\{\tfrac{1}{2}[B_2^{(T)}]_{S+N}\right\}^{-1/2} \qquad (63a)$$

$$z_T^{(\beta)} = \left\{\log\left(\mathcal{K}/\mu\right) - \frac{1}{2}\sum_{m=2}^{\infty}\frac{(-1)^m}{m}a_0^{2m}[B_m^{(T)}]_D\right\}$$
$$\cdot\left\{\tfrac{1}{2}[B_2^{(T)}]_D\right\}^{-1/2}, \qquad (63b)$$

provided $(a_0^2 \lesssim 1)$; [the strict condition for convergence is given in (55)]. The Bayes risk then follows from (7).

This completes our present exposition of the general case of colored noise signals in colored noise backgrounds. We shall apply this in the succeeding sections to the important class of problems where the noise accompanying the signal is white. Explicit determination of the iterated kernels $[B_m^{(T)}]_D$ and the eigenvalues $[\lambda_j^{(T)}]_D$, etc., are reserved for a later study.

V. Optimum Threshold Reception in Stationary White Noise (Continuous Sampling)

A. Detector Structure

Generally, the most common situation in communications practice occurs when the noise background is supplied for the most part by the shot and thermal noise of the receiver itself, and so is essentially white,

from the spectral point of view, and normal statistically. Stationarity is also a valid assumption in most instances, as long as the observation period $(0, T)$ is not excessively long. In examining the important cases of threshold reception, using the trace method where continuous sampling procedures are employed, we shall begin with detector structure.

Here it is convenient to let $\boldsymbol{\psi}_N^{-1}\zeta(t) = z_T(t)$, $(0 \leq t \leq T)$, in (51b), and observe for stationary white noise backgrounds that

$$K_N(t, u) = K_N(|\,t - u\,|) = \frac{W_{0N}}{2}\delta(t - u), \qquad (64)$$

where W_{0N} is the spectral density of this noise process, so that the solution of the first integral (51a) is at once $x(t) = (2/W_{0N})V(t)$, $(0 \leq t \leq T)$. The second integral (51b) becomes accordingly

$$\int_0^T\left[K_S(t, u) + \frac{W_{0N}}{2}\delta(u - t)\right]z_T(u)\,du$$
$$= \frac{2}{W_{0N}}\int_0^T V(u)K_S(t, u)\,du, \qquad (0 \leq t \leq T), \qquad (65)$$

whose solution, when the signal process is stationary and has rational spectra, is given in general terms by (231b) with (237)-(239). The resulting $z_T(t)$, $(0 \leq t \leq T)$, however, is a linear functional of the received data $V(t)$, and although systems can be built to compute the structure term Φ_T, (52a), *viz.*,

$$\Phi_T = \int_0^T V(t)z_T(t)\,dt, \qquad (66)$$

it is more convenient from the point of view of system design, as we shall see presently, to obtain an alternative form of solution that is independent of the received wave.

This is easily done, if we return for the moment to the matrix forms (43)-(49) characteristic of discrete sampling. We now regard $\boldsymbol{\psi}_N$ as the mean intensity $W_{0N}B$ of *band-limited* white noise, letting $B \rightarrow \infty$ at a suitable stage of the analysis, so as to give us once again the white noise background of our present problem. Now for this band-limited white noise we observe that its covariance function is

$$K_N(|\,t_1 - t_2\,|) = \psi_N\frac{\sin 2\pi B(t_1 - t_2)}{2\pi B(t_1 - t_2)}, \qquad (67)$$

so that if data is sampled at the times $t_j = j/2B(j = 1, \cdots, n)$ at which $K_N(|\,t_i - t_j\,|)$ is zero $(i \neq j)$, we have $\mathbf{K}_N = \psi_N\mathbf{I}$. Consequently, (43) becomes

$$\mathbf{X} = \psi_N^{-1}\mathbf{IV}. \qquad (68)$$

Letting $(\psi_N\mathbf{C})_{ii} = \rho_T(t_i, t_i)\Delta t$, [cf. (259)], and using (47) in (49) we get finally

$$\mathbf{K}_S = (\boldsymbol{\Lambda}_N + \mathbf{K}_S)\psi_N\mathbf{C}, \quad \text{or} \qquad (69a)$$

$$(\mathbf{K}_S)_{ik} = \sum_i^n\left[\frac{W_{0N}}{2}\frac{n}{T}\delta_{ii} + (\mathbf{K})_{ii}\right]\rho_T(t_i, t_k)\Delta t, \qquad (69b)$$

here we have made use of the fact that $2BT = n$, as $T \to \infty$ in

$$\psi_N = W_{0N} B; \lim_{B \to \infty} \psi_N = \lim_{n \to \infty} \frac{W_{0N} n}{2T}.$$

Passing to the limit then yields the desired integral equation in ρ_T, viz,

$$\int_0^T \left[K_S(t, u) + \frac{W_{0N}}{2} \delta(u - t) \right] \rho_T(\tau, u) \, du$$

$$= K_S(t, \tau), \qquad (0 \le t, \tau \le T), \qquad (70)$$

which is one form of a somewhat more general relation obtained by Price.[14] (For further details, see Appendix IV-E.) Since $\mathbf{Z} = \Psi_N \mathbf{CV}$, (47), the continuous version is [vide (262)],

$$Z(t) = \int_0^T V(t') \rho_T(t, t') \, dt', \qquad (71)$$

where $\rho_T(t, t') = \rho_T(t', t)$, (261). From (263) and (268) we find that the solutions of the two integral equations (55) and (70) are related through (71), where now[15]

$$Z(t) = \frac{W_{0N}}{2} z_T(t) \qquad (0 \le t \le T), \qquad (72)$$

and that the structure factor Φ_T can be written, alternatively to (66), as

$$\Phi_T = \frac{2}{W_{0N}} \iint_0^T V(t_1) \rho_T(t_1, t_2) V(t_2) \, dt_1 \, dt_2. \qquad (73)$$

Here ρ_T is indeed independent of the data $V(t)$. Solutions of (70) may be obtained by the methods of Appendix IV-A; as noted therein for the particular kernels involved here, we can find the solution at once by inspection of (231b), with the aid of (71) and (72). The results for general, rational signal spectra are given by (269). In the special cases or RC and LRC-noise signals (Appendix III-D and III-E) explicit solutions follow at once from Appendix IV-C and D as applied to (269), *et seq.*

While Φ_T can always be computed directly for a particular $V(t)$, it is possible to put it in a form that is easier for the system designer to work with. There are many ways of setting up Φ_T, (66), (73), for this purpose, two of which we shall describe below briefly. We note first that since the integrand of (73) is symmetric, *vide* (261), we may make use of the following relation, as employed originally by Price in a similar connection,[16]

$$\int_0^T \int_0^T A(x, y) \, dx \, dy = 2 \int_0^T dx \int_0^y A(x, y) \, dy,$$

$$A(x, y) = A(y, x) \qquad (74)$$

to write

$$\Phi_T = \frac{2}{W_{0N}} \int_0^T V(t) \, dt \int_0^t V(\tau) \rho_T(t - \tau; t) \, d\tau. \qquad (75)$$

Since we may set $\rho_T(t, \tau) = \rho_T(t - \tau, t)$, etc. equal to zero for all $t - \tau < 0$ without affecting Φ_T, we can interpret $\rho_T(t - \tau, t)$ as the weighting function (*i.e.*, impulse response for a unit impulse applied at time t) of a physically realizable, time-varying, linear filter. Thus,

$$z(t) = \int_0^t V(\tau) \rho_T(t - \tau, t) \, d\tau$$

is the output of this filter at time t, and Φ_T can be resolved into a sequence of realizable linear and nonlinear operations, as indicated in Fig. 1. The data $V(t)$
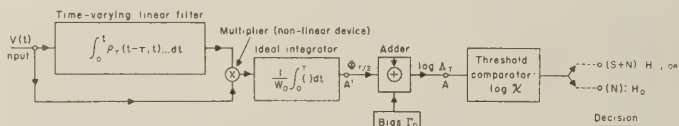
Fig. 1—Schematic diagram of one possible optimum receiver operation for the detection of a (normal) noise-signal in white, normal noise.

is passed into the time-varying filter, and then its output for all t is multiplied by the original input data $V(t)$. The product is next passed through an ideal filter [weighting $1/W_{0N}$ in $(0, T)$] to give us a number, $(\frac{1}{2})I_T$, at the end of the interval $(0, T)$. Combined with the bias Γ_0, the result is $\log \Lambda_T$, which is next compared with the present threshold $\log \mathcal{K}$, (6), and a decision is made: "signal, as well as noise," if $\log \Lambda_T \ge \log \mathcal{K}$, or "noise alone," if $\log \Lambda_T < \log \mathcal{K}$. Observe here that multiplication is a nonlinear operation.

This representation of Φ_T in terms of a time-varying filter, multiplier, and integrator, however, is not unique. If we impose again the condition that our linear filter be made up of physically realizable networks, and add now the further constraint that the nonlinear operation (analogous to multiplication above) be performed by a *zero-memory* nonlinear element—here, a full-wave quadratic rectifier—we find (Appendix V) that Φ_T can be computed in terms of a "matched," linear, time-invariant filter,[17] followed by this zero-memory square-law detection,
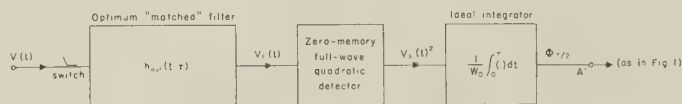
Fig. 2—Alternative system for the optimum detection of normal noise in white noise backgrounds.

and the same ideal integrator, see Fig. 2. We define a matched filter here as one that reduces the subsequent nonlinear operation for Φ_T to a zero-memory operation

[14] Bibliography [4], Part I, (1.54).
[15] The relationship (72) may also be established if we observe that $dZ'/dt = z_T(t)$, where $\mathbf{Z}' = \mathbf{CV} = \mathbf{Z}\psi_n^{-1}$, upon suitable passage to the limit.
[16] Bibliography [4], (29).

[17] This is a generalization of the matched predetection filter introduced by Van Vleck and Middleton [8] in their study of the reception of pulsed radar.

(Appendix V), *with minimum average cost of decision.* Thus Φ_T becomes, alternative to (66), (73), and (75)

$$\Phi_T = \frac{2}{W_{0N}} \int_0^T V_F(t)^2 \, dt, \qquad (76)$$

where $V_F(t)$ is the output in the interval $(0, T)$.

$$V_F(t) = \int_0^T V(t')h_{\text{opt.}}(t - t')_R \, dt', \quad (0 \le t \le T), \quad (77)$$

of the matched filter, when the input is switched on at $t = 0$ and off at $t = T$. Here $h_{\text{opt.}}(t - t')_R$ is the weighting function of this physically realizable, optimum predetection filter, whose structure is determined from the solution of the *nonlinear* integral (279), namely,

$$\rho_T(t, \tau)_{R\text{-opt.}} = \int_0^T h_{\text{opt.}}(x - t)_R h_{\text{opt.}}(x - \tau)_R \, dx,$$

$$(0 \le t, \tau \le T). \qquad (78)$$

This solution follows directly upon a double Fourier transform of both sides of (78), if we note that $\rho_T(t, \tau)_{R\text{-opt.}}$ vanishes outside the square $(0 \le t, \tau \le T)$, (70)—as a consequence of the condition that only operation on data in $(0, T)$ can here influence the simple binary decision process, at the end of this interval. The modulus of the system function of this optimum, matched filter is given generally by (cf. Appendix V)

$$| Y_{\text{opt.}}(i\omega)_R |$$

$$= T^{-1/2}\left\{ \iint_0^T \rho_T(t, \tau)_{R\text{-opt.}} e^{-i\omega(t-\tau)} \, dt \, d\tau \right\}^{1/2}, \qquad (79a)$$

$$= T^{-1/2}\left\{ \iint_0^T \rho_T(t, \tau)_{R\text{-opt.}} \cos \omega(t - \tau) \, dt \, d\tau \right\}^{1/2}. \qquad (79b)$$

An infinite number of matched filters is seen to be available, with weighting and system functions subject to (79). This is the result of fact that the observation process is incoherent, so that *a priori* phase information about the input is lacking. We cannot, therefore, expect to specify our predetector filter any more precisely, as the analysis shows. We have, in effect, an additional degree of freedom from the design viewpoint, since we are now at liberty to select those phase responses, $\phi_{\text{opt.}}(i\omega)_R$, in $Y_{\text{opt.}}(i\omega)_R = |\, Y_{\text{opt.}}(i\omega)_R \,| \exp [i\phi_{\text{opt.}}(i\omega)_R]$, which are most convenient to the particular system at hand.

We remark, finally, that Φ_T can be expressed in still other, more complicated ways. These representations are not unique, although our two examples above appear to be the simplest choices available. In any case, the decision as to what structure should be used and what elements chosen is left to the system designer's discretion, and will depend on how thoroughly and how cheaply he wishes to approach an ideal design. We shall refer to this point again, briefly in Section VIII, following, when we consider the question of suboptimum systems.

B. The Bias Γ_0

For continuous sampling in $(0, T)$, with a white noise background and threshold observation, the bias (199) becomes[18]

$$\Gamma_0 = \log \mu + \sum_{m=1}^{\infty} \frac{2^{m-1}(-1)^m}{m} \cdot \sigma_e^{2m}$$

$$\cdot \lambda^m (T^{-m}[B_m^{(T)}]_S), \qquad \lambda \equiv \omega_F T, \qquad (80)$$

when $[B_m^{(T)}]_S$ is the iterated kernel, order m, for $k_S(t, u)$ (195) and $\sigma_e^2 = \psi_S/W_{0N}\omega_F$, where ω_F is an angular frequency measure of the spectral width of the signal process and $\lambda (\equiv \omega_F T)$ is a measure of the duration of the observation period, in terms of the bandwidth of the signal. Thus, σ_e^2 is an *effective* input signal-to-noise power ratio.
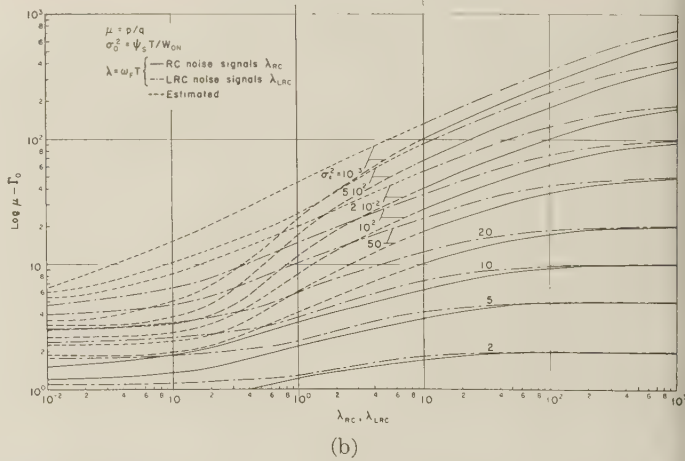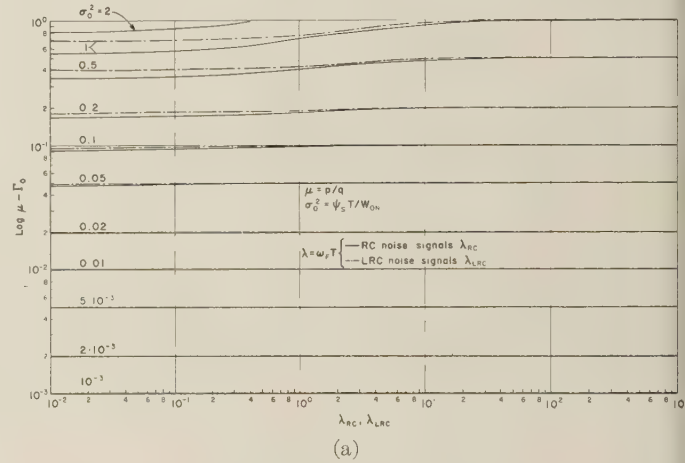
(a)

(b)

Fig. 3—(a) Bias (Γ_0) as a function of observation time and input signal energy for broad-band (RC) and narrow-band (LRC) noise signals in white background noise—continuous sampling. (b) Bias (Γ_0) as a function of observation time and input signal energy for broad-band (RC) and narrow-band (LRC) noise signals in white background noise-continuous sampling.

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[18] An exact expression for Γ_0, all σ_e^2, is given by $\log \mu$

$$+ \log \prod_{j=1}^{\infty} \left\{ 1 + \left(\frac{2\psi_S}{W_{0N}} \right)^m [\lambda_j^{(T)}]_S \right\}^{-1/2},$$

on comparing (199) with (161). Here $2\psi_S/W_{0N} = 2\sigma_e^2\omega_F$, and the $[\lambda_j^{(T)}]_S$ are the eigenvalues of (152a), when $G(t, \tau)$ therein is replaced by $K_S(1t - \tau1)$. For solutions in the case of RC-noise signals, see Table I and Price's explicit calculations of eigenvalues; Bibliography [4], Part II; also Report 30-40, p. 15.

here $W_{0N}\omega_F \equiv (\psi_N)$ is proportional to the average noise power passed by the predetector, matched filter. Here *ide* Appendix III and Fig. 3) the results are expressed in rms of the total input signal-to-noise energy $\sigma_0^2 \equiv$ T/W_{0N} accepted in $(0, T)$.

Fig. 3, p. 96, shows the bias (actually $\log \mu - \Gamma_0$) for a de range of values of input signal-to-noise ratio and tegration times for RC-noise signal (*i.e.*, a broad-band gnal), and the case of high-Q, (*i.e.*, narrow-band) LRC-oise signals. (Note that $(\omega_F)_{RC} \neq (\omega_F)_{LRC}$, (208c) *et. seq*). ne calculations are based on approximations of the iterated rnels $[B_m^{(T)}]_S$, as described in Appendix III-D and III-E here the various bias terms are given in (206a), (206b), 11a), and (211b), for a number of extreme conditions.

which has the formal advantage of the indicated symmetry with respect to $\mathcal{K}/\mu$. Here $R^*(\infty)$ is the Bayes risk for infinite input signal-to-noise ratios, while $\mathcal{R}_0$ is the irreducible average risk, and $R^*(0)$ is $qC_{1-\alpha} + pC_\beta$ if $\mathcal{K}/\mu < 1$, and is $qC_\alpha + pC_{1-\beta}$ when $\mathcal{K}/\mu > 1$.

By specializing the results of Section IV to white noise backgrounds, and in particular with the help of (62)-(63) and the various expressions for the iterated kernels in Appendix III, we see again for threshold reception that

$$\alpha^* \simeq \tfrac{1}{2}\{1 - \Theta[z_T^{(\alpha)}/\sqrt{2}]\};$$
$$\beta^* \simeq \tfrac{1}{2}\{1 + \Theta[z_T^{(\beta)}/\sqrt{2}]\}, \tag{83}$$

where now specifically for the RC- and high-Q, LRC-noise signals of Appendix III-D and III-E, we have

$$(z_T^{(\alpha)})_{RC} \approx \frac{\log (\mathcal{K}/\mu) + (\lambda/2)[\sqrt{1 + 4\sigma_e^2} - 1] - \lambda\sigma_e^2(1 + 4\sigma_e^2)^{-1/2}}{\sigma_e^2 \sqrt{2\lambda}/(1 + 4\sigma_e^2)^{3/4}} ; \qquad \sigma_e^2 \geq 0; \qquad \lambda > 5 \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right] \cdot \tag{84a}$$

$$(z_T^{(\beta)})_{RC} \approx \frac{\log (\mathcal{K}/\mu) + (\lambda/2)[\sqrt{1 + 4\sigma_e^2} - 1] - \sigma_e^2\lambda}{\sigma_e^2 \sqrt{2\lambda}} , \qquad \lambda = \lambda_{RC} = (\omega_F)_{RC}T. \tag{84b}$$

proximate as these results are, they are remarkably ose to the results obtained by using the exact develop-ent in terms of eigenvalues.[19] Comparison with Price's sults[18] indicates that for strong-signals [the most nfavorable case, since the series (80) is then poorly nvergent, if at all] our approximations are about per cent too large ($\sigma_0^2 = 20$ db), while for the longer tegration times and weaker signals, they err by 0.1 r cent or less in many instances. We conclude, then, at certainly in the threshold situations our approximate eatment, using the trace-method, gives satisfactory sults if not too high a degree of accuracy is required.

Bayes Risk and Minimum Detectable Signal

The Bayes risk is given by (7), a convenient form of hich is the normalized expression

For the usual situation of weak signals ($\sigma_e^4 \ll 1$) and long integration times ($\lambda \gg 1$) these reduce readily to the simpler expressions

$$(z_T^{(\alpha)})_{RC} \doteq \frac{1}{\sqrt{2}} \left[\frac{\log \left(\frac{\mathcal{K}}{\mu}\right)}{\sigma_e^2 \sqrt{\lambda}} + \sigma_e^2 \sqrt{\lambda}\right];$$

$$(z_T^{(\beta)})_{RC} \doteq \frac{1}{\sqrt{2}} \left[\frac{\log \left(\frac{\mathcal{K}}{\mu}\right)}{\sigma_e^2 \sqrt{\lambda}} - \sigma_e^2 \sqrt{\lambda}\right]. \tag{85}$$

When these relations are inserted into (83), we have the continuous analogs of earlier results for a simple optimum, pulsed radar.[20] As expected for threshold reception, the Bayes risk (82) is a simple function of $\sigma_e^2 \sqrt{\lambda}$ under these conditions.

For the narrow-band, LRC-noise signal, Appendix III-E. we get similar results:[21]

$$(z_T^{(\alpha)})_{LRC} \approx \frac{\log (\mathcal{K}/\mu) + (\lambda/4)[\sqrt{1 + 4\sigma_e^2} - 1] - (\lambda/2)\sigma_e^2(1 + 4\sigma_e^2)^{-1/2}}{\sigma_e^2 \sqrt{\lambda}(1 + 4\sigma_e^2)^{-3/4}} \doteq \left(\frac{\log (\mathcal{K}/\mu)}{\sigma_e^2 \sqrt{\lambda}} + \frac{\sigma_e^2 \sqrt{\lambda}}{2}\right) \tag{86a}$$

$$*_N = \frac{R^* - \mathcal{R}_0}{p(C_\beta - C_{1-\beta})} \equiv \frac{\mathcal{K}}{\mu} \alpha^* + \beta^*,$$

$$\mathcal{R}_0 = qC_{1-\alpha} + pC_{1-\beta}. \tag{81}$$

nother normalization is [9]

$$*_N = \frac{R^* - R^*(\infty)}{R^*(0) - R^*(\infty)} = \frac{\mathcal{K}}{\mu} \alpha^* + \beta^*, \quad \frac{\mathcal{K}}{\mu} > 1 \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\}, \tag{82}$$

$$= \alpha^* + \frac{\mu}{\mathcal{K}} \beta^*, \quad \frac{\mathcal{K}}{\mu} < 1$$

$$(z_T^{(\beta)})_{LRC} = \frac{\log \frac{\mathcal{K}}{\mu} + \frac{\lambda}{4}\left(\sqrt{1 + 4\sigma_e^2} - 1\right) - \sigma_e^2\lambda/2}{\sigma_e^2 \sqrt{\lambda}}$$

$$\doteq \left[\frac{\log \left(\frac{\mathcal{K}}{\mu}\right)}{\sigma_e^2 \sqrt{\lambda}} - \frac{\sigma_e^2}{2} \sqrt{\lambda}\right], \tag{86b}$$

where now $\lambda = \lambda_{LRC} = (\omega_F)_{LRC}T$. The correction terms to the error probabilities (189), (189a), indicate that λ should be $0(10^2)$, or more, for reasonable accuracy.

Curves of normalized Bayes risk are shown in Fig. 4. They exhibit the characteristic falling-off to zero as λ,

[19] In certain special cases, for example, when the kernel of the egral equation has the form $e^{-\alpha|t|}$, it is possible to obtain closed ms for the bias, without having to calculate eigenvalues. See particular, Price, Bibliography [4], PGIT paper, footnote 26. e method is due to Siegert [19]. (This is, however, not possible the error probabilities.)

[20] Bibliography [1], Example 1, (3.9).
[21] We replace λ_{RC} in (84a), (84b), and (85) by $\lambda_{LRC}/2$, when σ_e is used, or by $2\lambda_{LRC}$ when σ_0 appears; (213), also (210), (211).

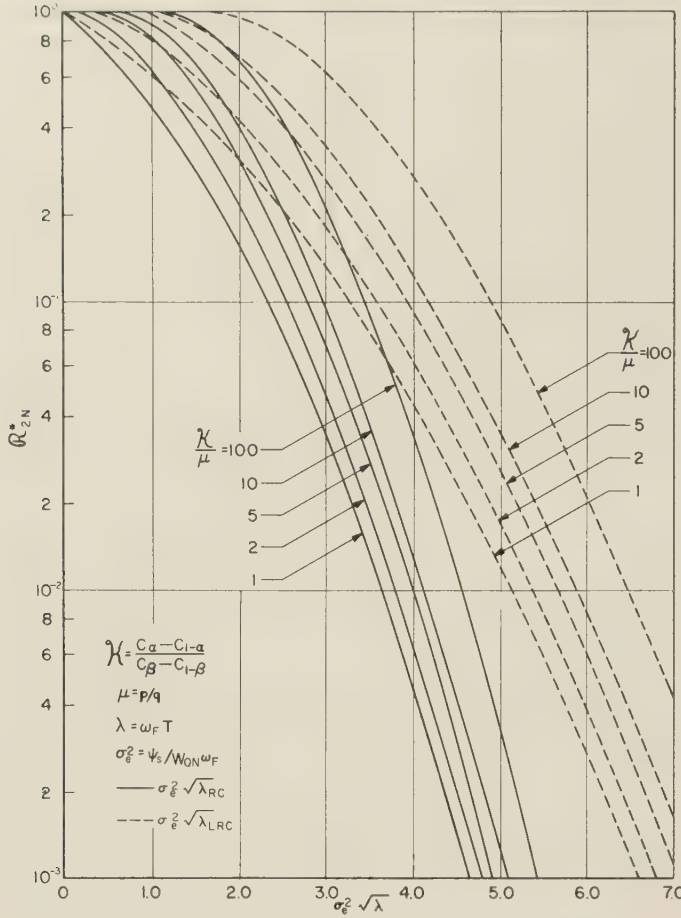Fig. 4—Normalized Bayes risk $\mathcal{R}_{2N}^*$ as a function of $\sigma_e^2 \sqrt{\lambda}$ for broad-band (RC) and narrow-band (LRC) noise signals in a white noise background-threshold reception and continuous sampling.

or σ_e^2, becomes indefinitely large. Depending on the value of $\log \mathcal{K}/\mu$, $(\gtrless 0)$, these curves will lie above or below that for $(\mathcal{K}/\mu = 1)$.

Besides the Bayes risk and the error probabilities themselves, a quantity of related interest is the *minimum detectable signal*, defined as that value of input signal (power) which will yield a certain (Bayes) average risk at the output of the decision system when the background noise level is specified and a threshold $\mathcal{K}$ is chosen. Here we shall define $(a_0^2)_{min}$ [or for continuous sampling, $(\sigma_e^2)_{min}$] as that value of the input which, on average performance, yields some specified portion (η) of the *maximum, normalized* Bayes risk $\mathcal{R}_{2N}^*$. The fractions chosen here are $\eta = 0.1$, 0.01, and 0.001. Thus, from the calculations of $\mathcal{R}_{2N}^*$ we obtain directly

$$(\sigma_e^2)_{min-\eta} = M_0 / \sqrt{\lambda}, \qquad (87)$$

where M_0 is a number determined from the value of $(\sigma_e^2)_{min-\eta} \sqrt{\lambda}$ in $\mathcal{R}^*$ that yields $\eta \mathcal{R}^*(0) = \eta \mathcal{R}_{max}^*$; (the maximum average risk occurs, of course, for $\sigma_e^2 = 0$.) The significant feature of this result is that $(\sigma_e^2)_{min}$ is inversely proportional to the *square-root* of the observation time, for threshold reception, as the curves of Fig. 4 show. [For very strong signals, on the other hand, as the results of Section VI indicate, $(\sigma_e^2)_{min-\eta} \backsim \lambda^{-1}$, characteristic

of all such cases, whether coherent or not, when the background noise is effectively suppressed.] It should be emphasized that there is nothing particularly advantageous in defining minimum detectable signal in terms of this particular, normalized Bayes risk (82) apart from a certain general analytical convenience. Other normalizations, or none at all, may serve as well, with $(\sigma_e^2)_{min}$ defined similarly. The procedure here, like that of system design, is left open for the particular application at hand.[22]

D. Minimax Detection

So far we have assumed that the *a priori* probabilities (p, q) that a data sample $V(t)$ does, or does not, contain a signal are known. In many cases, however, (p, q) are not known to the observer, and it is then a natural question to ask how one would operate an optimum receiver when this information is unavailable. One way of protecting ourselves against ignorance is to seek the "worst-best" operation, or more technically, a *Minimax* system.[23] Although Minimax operation can be overconservative, it does guard against the worst possibilities, *on the average*. Let us, accordingly, find the Minimax system for the present problem of optimum threshold detection.

We make use now of the fact that a Minimax system is one for which the maximum conditional risk[24] is never greater than the maximum conditional risks of other systems, for all possible signals. We use this fact[25] and (83) for these weak signal cases to write finally as our Minimax relation the Minimax probabilities

$$\frac{C_{1-\alpha} - C_{1-\beta} + C_\alpha - C_\beta}{C_\beta - C_{1-\beta}}$$
$$= \Theta(z_M^{(\beta)*}/\sqrt{2}) + \mathcal{K}\Theta(z_M^{(\alpha)*}/\sqrt{2}), \qquad (88)$$

which determines p_M^* and q_M^*. When this has solutions $\mu_M^* (= p_M^*/q_M^*)$ is determined, with $p_M^* = 1 - q_M^*$, of course.

As an example, let us take $C_{1-\alpha} = C_{1-\beta} = 0$, and let us set $\mathcal{K} = 1$.[26] Then, $C_\alpha = C_\beta (> 0)$, and from (88) we get directly, for both the RC and LRC cases above,

$$\log \mu_M^* = 0. \qquad (89)$$

The Minimax system is still our Bayes system with $p_M^* = q_M^* = 1/2$ now. Thus, when p and q are unknown to the observer, and he *sets* them equal, so that $\log \mu = 0$, he is operating a Minimax detection system. He is guarding against the worst case, *i.e.*, the most expensive errors, *as is the case with Price's system [4] under these conditions*. Of course, when $p = 1 - q$ is available to the observer, then he should make full use of this information, as done above. The Minimax average risk R_M^* always exceeds (or at best, is equal to) the Bayes risk R^*, and so by comparing $\mathcal{R}_{2N}^*$, $(\mathcal{R}_{2N}^*)_{min-max}$ here, we can form some idea

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[22] Bibliography [1], (3.6).
[23] For a detailed discussion, see Bibliography [1], (2.10) and for detection, in particular, Bibliography [1], (3.4).
[24] Bibliography [1], (3.4).
[25] Bibliography [1], (3.9), and (3.20).
[26] The ideal observer condition, Bibliography [1], (3.3).

the price we must pay for our ignorance of these *a priori* probabilities.

The Minimax average risk in our present condition of continuous sampling and threshold reception is determined at once from (89) [$\mathcal{K} = 1$, here] in (83) *et seq.* We have

$$\alpha_M^* = \tfrac{1}{2}\{1 - \Theta(x/\sqrt{2})\} = \beta_M^*, \qquad (90)$$

where $x = x_{RC} = (\sigma_e^2)_{RC} \sqrt{\lambda_{RC}/2}$, or $x = x_{LRC} = (\sigma_e^2)_{LRC} \sqrt{\lambda_{LRC}/2}$ for our RC- or (high-Q) LRC-noise signals, under the special cost assignments introduced above. The Minimax average risk is

$$R_M^*/C_\beta = \alpha_m^* = \beta_m^*$$
$$= \tfrac{1}{2}\{1 - \Theta(x/\sqrt{2})\}, \qquad [\mathcal{K}, \mu_M^* = 1]. \qquad (91)$$

A typical Minimax average risk curve is given in Fig. 4 for $\mathcal{K} = 1$, *and* $\mu = 1$. Minimum detectable signals in the Minimax situation are determined exactly as in the Bayes cases above. We have $\eta(R_M^*/C_\beta)_{\max} = \eta/2$, where $(0 < \eta < 1)$ is the fractional level at which $(\sigma_e^2)_{\min-\eta}^{(M)}$ is to be calculated, so that here

$$(\sigma_e^2)_{\min-\eta}^{(M)} = 2\Theta^{-1}(1 - \eta)/\sqrt{\lambda_{RC}}, \qquad \text{or} \quad \Bigg\} \qquad (92)$$
$$= 2\sqrt{2}\Theta^{-1}(1 - \eta)/\sqrt{\lambda_{LRC}}, \qquad (93)$$

respectively for the broad-band RC-noise signal and for the narrow-band LRC-noise signal. Similar calculations for other cost assignments may be effected in the same way, with the aid of (88).

VI. A Special Case: Discrete, Independent Sampling

A problem of some practical interest arises when the sampling process is made discrete, with intervals between sampling times that are much longer than the average fluctuation time of the signal and background noise processes. Then both $\mathbf{K}_S$ and $\mathbf{K}_N$ are essentially diagonal matrices, *e.g.*, $\mathbf{K}_S = \psi_S \mathbf{I}$, $\mathbf{K}_N = \psi_N \mathbf{I}$, and we have a case of *random sampling*. From the analytical point of view this is interesting, since it is a situation where comparatively simple exact expressions for the bias and error probabilities can be found, and since it represents one limiting form of data processing, as opposed to the other extreme of continuous sampling. Thus, by comparing Bayes risks, or minimum detectable signals, for a common threshold in the two cases, we can gain some idea of what we lose in system sensitivity when the information inherent in the correlated sample [*i.e.*, the continuous data in $(0, T)$] is thrown away.

Optimum structure, bias, and average performance are now easily determined. From (19a) we see that

$$\mathbf{C} = \psi_N^{-1}\mathbf{I} - (\psi_S + \psi_N)^{-1}\mathbf{I}$$
$$= \psi_N^{-1}\left(\frac{a_0^2}{a_0^2 + 1}\right)\mathbf{I}; \qquad a_0^2 \equiv \psi_S/\psi_N. \qquad (94)$$

The structure factor Φ_n, (22), is now

$$\Phi_n = \psi_N^{-1}\tilde{\mathbf{V}}\mathbf{V}\left(\frac{a_0^2}{1 + a_0^2}\right) \qquad (95)$$

while the bias, Γ_0, of (23) reduces to

$$\Gamma_0 = \log \mu - \frac{n}{2} \log (1 + a_0^2). \qquad (96)$$

The error probabilities follow directly from (32a), (32b), and (36). Thus, for the optimum detector embodied in (23), we have now

$$\log \Lambda_n = \log \mu - \frac{n}{2} \log (1 + a_0^2) + \frac{1}{2}\left(\frac{a_0^2}{1 + a_0^2}\right)\tilde{\mathbf{v}}\mathbf{v}; \qquad (97)$$

we find first that the distribution densities of $\log \Lambda_n$ become [(32a), (32b), and (94), etc.], with $B_0 = a_0^2/(a_0^2 + 1)$,

$$Q_n(x) = \int_{-\infty}^{\infty} \frac{e^{-i\xi(x - \Gamma_0)}}{(1 - i\xi B_0^2)^{n/2}} \frac{d\xi}{2\pi}$$
$$= B_0^{-n/2}\Gamma(n/2)^{-1}(x - \Gamma_0)^{(n/2-1)}$$
$$\cdot e^{-(x-\Gamma_0)/B_0}, \quad (x > \Gamma_0)$$
$$= 0, \qquad\qquad\qquad (x < \Gamma_0) \qquad (98)$$

and

$$P_n(x) = \int_{-\infty}^{\infty} \frac{e^{-i\xi(x - \Gamma_0)}}{(1 - i\xi a_0^2)^{n/2}} \frac{d\xi}{2\pi}$$
$$= a_0^{-n/2}\Gamma(n/2)^{-1}(x - \Gamma_0)^{(n/2-1)}$$
$$\cdot e^{-(x-\Gamma_0)/a_0^2}, \qquad (x > \Gamma_0)$$
$$= 0, \qquad\qquad\qquad (x < \Gamma_0). \qquad (99)$$

These are recognized as χ^2 densities, with n degrees of freedom [10]. Applying (98) and (99) to (9), we see that the error probabilities α^*, β^* may be expressed in terms of the incomplete Gamma function

$$I_c(x, \gamma) \equiv \Gamma(\alpha)^{-1} \int_0^x e^{-y}y^{\gamma-1} \, dy,$$
$$\Re e(\gamma) > 0, \qquad x > 0; = 0, \qquad x < 0. \qquad (100)$$

TABLE I

$\mathcal{K}/\mu$	$\log \dfrac{\mathcal{K}}{\mu}$	$a_0^2 \to 0; (n \geq 1)$			$a_0^2 \to \infty \ (n \geq 1)$		$a_0^2 > 0; (n \to \infty)$		
		α^*	β^*	$\Re^*_{in}$	α^*	β^*	α^*	B^*	$\Re^*_{in}$
> 1	> 0	0	1	1	0	0	0	0	0
$= 1$	$= 0$	$1 - I_c\left(\dfrac{n}{2}, \dfrac{n}{2}\right)$	$I_c\left(\dfrac{n}{2}, \dfrac{n}{2}\right)$	1	0	0	0	0	0
< 1	< 0	1	0	$\mathcal{K}/u$	0	0	0	0	0

We have specifically

$$\alpha^* = 1 - I_c\left[\frac{\log\,(\mathcal{K}/\Gamma_0)}{B_0}\;;n/2\right] \qquad (101a)$$

$$\beta^* = I_c[a_0^{-2}\log\,(\mathcal{K}/\Gamma_0);n/2], \qquad (101b)$$

and the Bayes risk $\mathcal{R}_{IN}^*$, (81), is simply

$$\mathcal{R}_{IN}^* = \frac{\mathcal{K}}{\mu}\,\alpha^* + \beta^* \qquad (102)$$

where the I_c in α^*, β^* are tabulated functions [11]. Values of α^*, β^*, and $\mathcal{R}^*$ are shown in Table I, p. 99.

Specifically, the arguments of I_c in (101) are respectively

$$\begin{aligned}
\frac{\log\,(\mathcal{K}/\Gamma_0)}{B_0} &= \frac{n}{2}\left(\frac{1+a_0^2}{a_0^2}\right)\log\,(1+a_0^2) \\
&\quad + \left(\frac{a_0^2+1}{a_0^2}\right)\log\frac{\mathcal{K}}{\mu}\,, \qquad (103a) \\
a_0^{-2}\log\,(\mathcal{K}/\Gamma_0) &= \frac{n}{2}\frac{\log\,(1+a_0^2)}{a_0^2} \\
&\quad + \frac{1}{a_0^2}\log\frac{\mathcal{K}}{\mu}\cdot \qquad (103b)
\end{aligned}$$

In the threshold cases, we may employ the techniques of Appendix II-C, to get the characteristic normal distribution densities for P_n, Q_n (with correction terms in the Edgeworth series). The error probabilities (101) are now approximately

$$\alpha^* \simeq \tfrac{1}{2}\{1 - \Theta(z_0')\};\qquad \beta^* \simeq \tfrac{1}{2}\{1 + \Theta(z_0'')\},$$

$$\text{all}\quad a_0^2 \geq 0,\qquad \text{large } n, \qquad (104)$$

where

$$z_0' \approx \frac{\sqrt{n}}{2}\left\{\left(\frac{a_0^2+1}{a_0^2}\right)\log\,(1+a_0^2) - 1\right\}$$

$$+ \log\frac{\mathcal{K}}{\mu}\bigg/\frac{a_0^2\sqrt{n}}{(a_0^2+1)} \doteq \frac{\log\frac{\mathcal{K}}{\mu}}{a_0^2\sqrt{n}} + \frac{a_0^2\sqrt{n}}{4}\,, \qquad (105a)$$
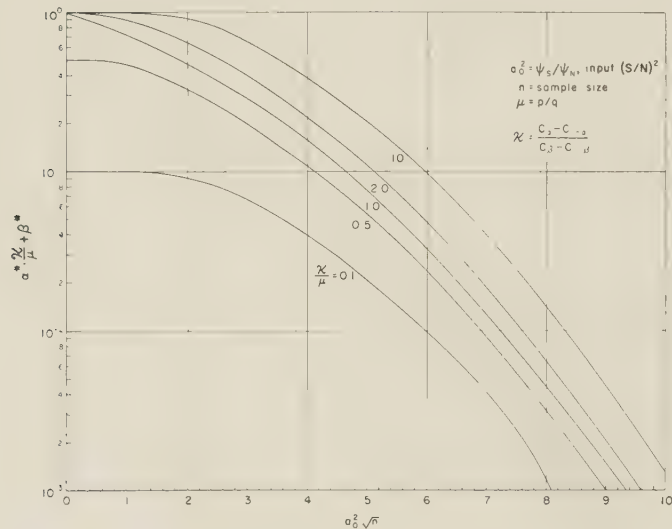
Fig. 5—Normalized Bayes risk $\mathcal{R}_{IN}^*$ as a function of $a_0^2\sqrt{n}$; threshold reception and discrete random sampling of colored noise signals in band-limited or colored noise.
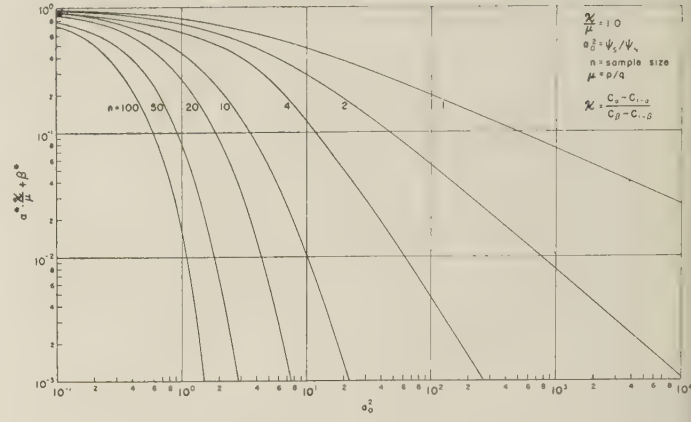
Fig. 6—Normalized Bayes risk $\mathcal{R}_{IN}^*$ as a function of input signal-to-noise ratio a_0^2. Discrete random sampling of colored noise signals in band-limited or colored noise.

$$z_0'' \approx \frac{n}{2}\left[\frac{\log\,(1+a_0^2)}{a_0^2} - 1\right] + \frac{\log\,(\mathcal{K}/\mu)}{a_0^2\sqrt{n}}$$

$$\doteq \frac{\log\frac{\mathcal{K}}{\mu}}{a_0^2\sqrt{n}} - \frac{a_0^2\sqrt{n}}{4}\,, \qquad (105b)$$

vide (85) and (86). Curves of Bayes risk and minimum detectable signals are shown in Figs. 5-7, based on (101) and (104), (105). Again observe the characteristic dependence of $(a_0^2)_{\min\text{-}\eta}$ on $n^{-1/2}$ for threshold detection. For large a_0^2, we get essentially $(a_0^2)_{\min\text{-}\eta} \frown n^{-1}$, as expected, while for intermediate values of the input signal-to-noise ratio the minimum detectable signal varies as η^ϵ, $(1/2 \leq \epsilon \leq 1)$.

VII. ALTERNATIVE APPROACH FOR NARROW-BAND SIGNALS

A. Preliminary Remarks

When the signal process is narrow-band we can obtain an alternative representation of optimum (and suboptimum) detection in terms of the slowly-varying components S_c, S_s of the possible signal S, where now

$$S(t) = S_c(t)\cos\omega_0 t + S_s\sin\omega_0 t, \qquad \omega_0 = 2\pi f_0, \qquad (106)$$

Fig. 7—Minimum detectable signals for the normalized Bayes risk $\mathcal{R}_{IN}$. Discrete random sampling of colored noise-signals in band-limited or colored noise.

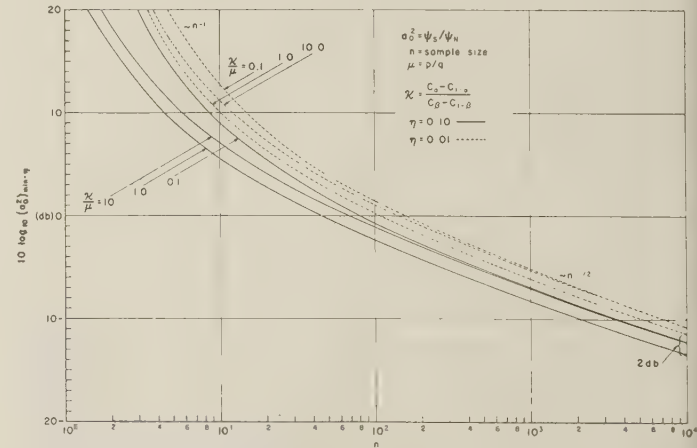nd f_0 is some characteristic frequency, usually the entral one in the signal spectrum, if it is symmetrical. For this discussion we shall assume that $S(t)$ does indeed possess a symmetrical intensity spectrum. The background noise process $N(t)$ can also be represented by

$$N(t) = N_c(t) \cos \omega_0 t + N_s(t) \sin \omega_0 t, \qquad (107)$$

where now $N(t)$ need not be narrow-band, although, of course, N_c and N_s are then no longer slowly-varying compared to $\sin \omega_0 t$, $\cos \omega_0 t$.

Let us examine the statistics of N_c and N_s, when N is normal. These components are also normal, ($\bar{N}_c = \bar{N}_s = \bar{N} = 0$), with covariance functions

$$\overline{N_c(t_i)N_c(t_k)} = \overline{N_s(t_i)N_s(t_k)} = \psi_N \rho_0(|t_i - t_k|) \qquad (108a)$$

$$\overline{N_c(t_i)N_s(t_k)} = -\overline{N_s(t_i)N_c(t_k)}$$
$$= \psi_N \lambda_0(t_i - t_k); \qquad \lambda_0(-t) = -\lambda_0(t), \qquad (108b)$$

where, in particular, $(t = t_i - t_k)$

$$\rho_0(t) \equiv \int_0^\infty \mathcal{W}_N(f) \cos(\omega - \omega_0)t \, df \Big/ \int_0^\infty \mathcal{W}_N(f) \, df, \qquad (109a)$$

$$\lambda_0(t) \equiv \int_0^\infty \mathcal{W}_N(f) \sin(\omega - \omega_0)t \, df \Big/ \int_0^\infty \mathcal{W}_N(f) \, df, \qquad (109b)$$

and $\mathcal{W}_N(f)$ is the intensity spectrum of $N(t)$. Now in the case of narrow-band noise we get the well-known result that, approximately

$$\rho_0(t) \doteq 2\psi_N^{-1} \int_0^\infty \mathcal{W}_N(f') \cos \omega' t \, df'; \qquad \lambda_0(t) \doteq 0, \qquad (110)$$

provided that $\mathcal{W}_N(f)$ is symmetrical about f_0, $f_0 \gg \Delta f$, the bandwidth of $N(t)$. Then N_c and N_s are essentially statistically independent, for all $t = (t_i - t_k)$, with a great simplification in the subsequent analysis.

For our present problem of white normal noise backgrounds, on the other hand, $N(t)$ is, of course, not narrowband, and moreover we do not expect that $N_c(t)$, $N_s(t)$ will be statistically independent for arbitrary noise bandwidths. However, in the limiting situation of white noise, as we shall show below, N_c and N_s, though rapidly varying, are independent, and we can make use of this fact to obtain (from the analytical point of view) a simpler system for detecting narrow-band signals than the general one described in Section V.

We start with band-limited white noise, for which $\mathcal{W}_N(f) = W_{0N}$, $0 < f < B$, $\psi_N = W_{0N}B$. Inserting this into (109a), (109b) we get

$$\rho_0(t) = \frac{\sin \omega_0 t + \sin 2\pi(B - f_0)t}{2\pi B t},$$

$$\lambda_0(t) = \frac{\cos \omega_0 t - \cos 2\pi(B - f_0)t}{2\pi B t}. \qquad (111)$$

Now observe that for $t \neq 0$, $\lim B \to \infty$, both $\rho_0(t)$, $\lambda_0(t)$ vanish, while if $t = 0$, we get

$$\lim_{B \to \infty} \rho_0(0) = 1, \qquad \lim_{B \to \infty} \lambda_0(0) = 0. \qquad (112)$$

From this, and the normal character of N_c and N_s, it follows that N_c and N_s are completely independent, (108a), (108b). Thus, we can use (107) to represent white noise (in the limit $B \to \infty$), and the "components" N_c and N_s are statistically unrelated for all f_0, t.

For $\lambda_0 \neq 0$, the distribution density of the n sampled values of N_c and N_s is

$$W_{2n}(\mathbf{N}_c, \mathbf{N}_s)_N = \frac{\exp\{-\frac{1}{2}\tilde{\mathbf{X}}_0 \mathbf{K}_0^{-1} \mathbf{X}_0\}}{(2\pi)^n (\det \mathbf{K}_0)^{1/2}},$$

$$\mathbf{X}_0 = \begin{bmatrix} \mathbf{N}_c \\ \mathbf{N}_s \end{bmatrix}; \qquad \bar{\mathbf{X}}_0 = 0, \qquad \mathbf{N}_c = [N_c(t_i)] \qquad (113)$$

and

$$\mathbf{K}_0 = \psi_N \begin{bmatrix} \varrho_0 & \lambda_0 \\ -\tilde{\lambda}_0 & \varrho_0 \end{bmatrix},$$

$$\text{where} \quad \varrho_0 = [\rho_0(|t_i - t_k|)] = \tilde{\varrho} \qquad (114)$$
$$\lambda_0 = [\lambda_0(t_i - t_k)] = -\tilde{\lambda}_0.$$

Writing $\mathbf{k}_0 \equiv \mathbf{K}_0 \psi_N^{-1}$, we see that $\det \mathbf{K}_0 = \psi_N^n \det (\varrho_0^2 + \lambda_0^2)$. However, for $\lambda_0 = 0$ we get

$$W_{2n}(\mathbf{N}_c, \mathbf{N}_s)_N = W_n(\mathbf{N}_c)_N W_n(\mathbf{N}_s)_N,$$

$$= \frac{\exp\{-\frac{1}{2}\tilde{\mathbf{N}}_c(\psi_N\varrho_0)^{-1}\mathbf{N}_c\}}{(2\pi)^{n/2}\sqrt{\det \psi_N\varrho_0}}$$

$$\cdot \frac{\exp\{-\frac{1}{2}\tilde{\mathbf{N}}_s(\psi_N\varrho_0)^{-1}\mathbf{N}_s\}}{(2\pi)^{n/2}\sqrt{\det \psi_N\varrho_0}}, \qquad (\psi_N < \infty), \qquad (115)$$

where these relations are appropriate to white noise in the limit $B \to \infty$. We can, nevertheless, use (113) as the distribution density of a set of noise components N_c, N_s, which become identical with the components of our actual white noise background in the limit. In fact, if B is at all large compared to the reciprocal of the sampling time $\delta(= t_{k+1} - t_k)$, (115) is an excellent approximation. We observe, finally, that since N_c, N_s (115) are independent (all t), and since it is assumed that signal and noise are also independent, we may write for the received wave

$$V(t) = V_c(t) \cos \omega_0 t + V_s(t) \sin \omega_0 t \qquad (116)$$

with the properties that for $V_c = N_c + S_c$, $V_s = N_s + S_s$ (if there is a signal),

$$\overline{V_{ci} V_{sk}} = 0, \qquad (117)$$

provided $\overline{S_{ci} S_{sk}} = 0$, *i.e.*, provided the spectrum of the signal is symmetrical. In this fashion V_c, V_s, for an additive signal, as well as noise, are statistically unrelated.

B. Structure of the Detector

The analysis of Section II-B is now applied to $\mathbf{V}_c$, $\mathbf{V}_s$, instead of $\mathbf{V}$ alone. The optimum detector $\log \Lambda_n(\mathbf{V})$ is replaced by

$$\log \Lambda_n(\mathbf{V}_c, \mathbf{V}_s)$$
$$= \log \mu + \log \{W_{2n}(\mathbf{V}_c, \mathbf{V}_s)_{S+N}/W_{2n}(\mathbf{V}_c, \mathbf{V}_s)_N\} \qquad (118)$$

where it is somewhat more direct to use the alternate approach implied by (18). For noise alone $(W_{2n})_N$ is just (115), with $\mathbf{N}_c$, $\mathbf{N}_s$ replaced by $\mathbf{V}_c$, $\mathbf{V}_s$, while

$$W_{2n}(\mathbf{V}_c, \mathbf{V}_s)_{S+N}$$

$$= \exp\{-\tfrac{1}{2}\tilde{\mathbf{V}}_c(\psi_N \varrho_0 + \mathbf{K}_{S_0})^{-1}\mathbf{V}_c$$
$$- \tfrac{1}{2}\tilde{\mathbf{V}}_s(\psi_N \varrho_0 + \mathbf{K}_{S_0})^{-1}\mathbf{V}_s\}$$
$$\cdot (2\pi)^{-n}\{\det(\psi_N \varrho_0 + \mathbf{K}_{S_0})\}^{-1}, \qquad (119)$$

since signal and noise are additive and independent. Here $\mathbf{K}_{S_0} = [\psi_S \rho_0(t_i - t_k)_S]$ is the slowly-varying part of the signal processes' covariance matrix, (109) with $\mathbb{W}_N(f)$ replaced by $\mathbb{W}_S(f)$. Writing

$$\mathbf{K}_{N_0} \equiv \psi_N \varrho_0 \qquad (120)$$

for the background noise, and applying (119) to (118), we get the alternative form of the optimum receiver structure in terms of the received data "components," V_c and V_s, and the slowly-varying part of the signal's covariance function. The result is

$$\log \Lambda_n(\mathbf{V}_c, \mathbf{V}_s) = \log \mu - \log \det(\mathbf{I} + \mathbf{K}_{S_0}\mathbf{K}_{N_0})$$
$$+ \tfrac{1}{2}\tilde{\mathbf{V}}_c\mathbf{C}_0\mathbf{V}_c + \tfrac{1}{2}\tilde{\mathbf{V}}_s\mathbf{C}_0\mathbf{V}_s \qquad (121)$$

where $\mathbf{C}_0 = \mathbf{K}_{N_0}^{-1} - (\mathbf{K}_{S_0} + \mathbf{K}_{N_0})^{-1}$, see (19a).

This expression is valid for discrete sampling where $\delta \gg B^{-1}$ and becomes precise in the limit $B \to \infty$ (T finite) of continuous sampling. From Section V we find that under these circumstances (121) reduces to

$$\log \Lambda_T(V_c(t), \qquad V_s(t)) = \Gamma_0'$$
$$+ \tfrac{1}{2}\Phi_T(V_c(t)) + \tfrac{1}{2}\Phi_T(V_s(t)) \qquad (122)$$

where Γ_0' is given by (201b) and Φ_T may be found from (66)' (73), (75), or (76), if we replace $V(t)$ therein respectively by $V_c(t)$ and $V_s(t)$. The actual operation of the detector is similar to that indicated in Figs. 1 or 2, except that a new bias, Γ_0', is used, and the Φ_T for both components are added together at A', before this bias is inserted. The same threshold, $\mathcal{K}$, is used, following A (Fig. 1).[27] The direct, or "broad-band" approach, described in Sections II-V, and the narrow-band or "componential" approach considered here, give the same performance when the thresholds are identical, as long as the input signal is a narrow-band process (observed in white noise). The simple structure of (122) breaks down when wideband signal processes are included.

This system (122) operates on the "components" V_c, V_s of the received wave V. However, V is certainly broad-band, since the accompanying noise is white, and the question arises as to how we obtain these components V_c, V_s analytically and physically so that they may be applied to our decision-making device. From the purely mathematical viewpoint, we can answer this question

using the Hilbert transform of V to write (approximately)

$$V_c(t) \doteq V(t)\cos\omega_0 t + \mathcal{H}(V)\sin\omega_0 t;$$
$$V_s(t) \doteq V(t)\sin\omega_0 t - \mathcal{H}(V)\cos\omega_0 t \qquad (123)$$

where

$$\mathcal{H}(V) = \frac{1}{\pi}\,\mathcal{P}\int_{-\infty}^{\infty}\frac{V(\tau)}{t-\tau}\,dt; \qquad \mathcal{H}^2(V) = -V \qquad (124)$$

and $\mathcal{P}$ denotes the principal part of the integral. Therefore, given V we can in principle calculate its Hilbert transform according to (124), and from (123) thus obtain the desired components V_c and V_s.

In practice, of course, this is almost always out of the question, but if the noise background is spectrally very broad compared to the signal, *i.e.*, $B \gg \Delta f_s$, yet narrow compared to the central frequency f_0 ($\gg B$), V_c and V_s *are* slowly-varying compared to $\cos\omega_0 t$, $\sin\omega_0 t$, and may be obtained from the received wave $V(t)$ by modulating it respectively by $\cos\omega_0 t$ and $\sin\omega_0 t$ and putting the result through a filter (of gain 2), whose upper cutoff frequency is less than $2f_0 - B_0$. This is a simple way of getting the desired components for use in (122), and at the same time is a reasonable practical approximation to the ideal situation and a very good approximation if B is much greater than signal bandwidth (as well as small compared to f_0).

C. *Error Probabilities*

Following the steps of Section III-A, we may write now

$$\alpha^* = \int_{\log\mathcal{K}}^{\infty} Q_n(y)\,dy; \qquad \beta^* = \int_{-\infty}^{\log\mathcal{K}} P_n(y)\,dy \qquad (125)$$

where

$$Q_n(y) = \iint_{-\infty}^{\infty} d\mathbf{V}_c\,d\mathbf{V}_s F(\mathbf{V}_c, \mathbf{V}_s \mid 0)$$
$$\cdot \delta(y - \log\Lambda_n[\mathbf{V}_c, \mathbf{V}_s]) \qquad (126a)$$

$$P_n(y) = \iint_{-\infty}^{\infty} d\mathbf{V}_c\,d\mathbf{V}_s\langle F(\mathbf{V}_c, \mathbf{V}_s \mid \mathbf{S}_c, \mathbf{S}_s)\rangle_S$$
$$\cdot \delta(y - \log\Lambda_n[\mathbf{V}_c, \mathbf{V}_s]). \qquad (126b)$$

Using (121) with (119) and (120) and the fact that $F_y(i\xi)_N$, $F_y(i\xi)_{S+N}$ follow from expressions like (28), (30), we get finally

$$F_y(i\xi)_N = e^{i\xi\Gamma_0'}\{\det(\mathbf{I} - i\xi\mathbf{k}_{N_0}\psi_N\mathbf{C}_0)\}^{-1} \qquad (127a)$$

$$F_y(i\xi)_{S+N} = e^{i\xi\Gamma_0'}\{\det[I - i\xi\psi_N(a_0^2\mathbf{k}_{S_0} + \mathbf{k}_{N_0})\mathbf{C}_0]\}^{-1} \qquad (127b)$$

where $\mathbf{k}_{N_0} = \varrho_0$ and $\mathbf{k}_{S_0} = \psi_0^{-1}\mathbf{K}_{S_0}$. The significant feature about these relations, compared with those obtained by the "broad-band" approach, (31), is that now the determinant appears to the first power (in the denominator), so that we may expect the exact treatment using the eigenvalue method to be effective here. The Fourier transform of these characteristic functions are calculated

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[27] Note that Γ_0', computed from (201b) for the high-Q, LRC noise signal (for which $K_{S_0}(t) \sim e^{-\omega_F t}$, formally identical with the RC case), is equivalently given by (192), (211a), and (211b), when (210) is employed.

(183) and (184), yielding $P_n(y)$, $Q_n(y)$. The error probabilities then follow according to (125). To illustrate here, we have, for only positive eigenvalues:

$$* \quad \sum_{k=1}^{N+} e^{-(\log \mathfrak{K} - \Gamma_0')/\lambda_{k,N}} \prod_{j=1}^{n+(j\neq k)} (1 - \lambda_j'/\lambda_k')_N^{-1},$$

$$\log \mathfrak{K} - \Gamma_0' \geq 0,$$

$$= \sum_{k=1}^{n+} \prod_{j}^{n+(j\neq k)} (1 - \lambda_j'/\lambda_k')_N^{-1},$$

$$\log \mathfrak{K} - \Gamma_0' < 0 \quad (128a)$$

and

$$* = \sum_{k=1}^{n+} [1 - e^{-(\log \mathfrak{K} - \Gamma_0')/\lambda_{k'},S+N}]$$

$$\prod_{j}^{n+(j\neq k)} (1 - \lambda_j'/\lambda_k')_{S+N}^{-1},$$

$$\log \mathfrak{K} - \Gamma_0' > 0$$

$$= 0 \quad , \log \mathfrak{K} - \Gamma_0' < 0 \quad (128b)$$

From the preceding analysis we note that $(\lambda_k')_N$, $(\lambda_k')_{S+N}$ are the eigenvalues of $\mathbf{G}_N'$, $\mathbf{G}_{S+N}'$, *i.e.*, the coefficients of $-i\xi$ in the determinants (127a), (127b) above, see (153). For continuous sampling in white noise, $(n +) \to \infty$, and the eigenvalues are found from the appropriate integral equations (152a).[28] In the threshold cases we can also use the trace-method, as was done in Section V, without having to calculate these eigenvalues directly.

VIII. Suboptimum Systems

The methods of decision theory are equally applicable to specified systems which are not optimum. By computing the average risk R, (3), in such cases, one can then compare the given (suboptimum) receiver with the optimum one for the same purpose, and in this way obtain a definite measure of the extent to which actual performance is degraded for the chosen criterion.[29] Here we shall briefly outline the analysis of a class of nonideal detection systems for operation in white noise backgrounds, to indicate how such comparisons might be made. This is a problem of considerable practical importance, since it is almost never possible to construct an exactly optimum system.

We let $\log L_n(\mathbf{V})$ be a general square-law system, since there is little point here in introducing a more complicated structure when it is known that the quadratic device with appropriate weighing is optimum, [see (21) and (23)]. Accordingly, let us write

$$\log L_n(\mathbf{V}) = G_0 + \tfrac{1}{2}\tilde{\mathbf{V}}\mathbf{H}\mathbf{V} = G_0 + \tfrac{1}{2}\Psi_n \quad (129)$$

where G_0 is the bias and $\mathbf{H}$ is the weighting, which in general is not equal to $\mathbf{C}$ for optimum receivers; G_0 may not be set equal to Γ_0, and usually we may wish to choose some $G_0 \neq \Gamma_0$ to help compensate for the fact that $\mathbf{H} \neq \mathbf{C}$.

Here Ψ_n, like Φ_n in (23), is the structure factor of the detector. Next, we let

$$\mathbf{H} = \psi_N^{-1}\mathbf{D}_0\mathbf{k}_N^{-1} \quad (130)$$

where now $\mathbf{D}_0$ is specified *a priori* in some manner and embodies the various filtering operations of the system, as we shall see presently. (For the moment the background noise is regarded as having a finite ψ_N — here as band-limited white noise. At the appropriate point we shall let $B \to \infty$ and use $\lim_{B\to\infty} \psi_N = \lim_{n\to\infty} nW_{0N}/2T$ in the analysis to obtain the desired expressions for continuous sampling in white noise backgrounds.) Again, unless $\mathbf{D}_0 = \mathbf{I} - (\mathbf{K}_s\mathbf{K}_N^{-1} + \mathbf{I})^{-1} = \mathbf{I} - (a_0^2\mathbf{D} + \mathbf{I})^{-1}$, (19a), our receiver is not optimum, *i.e.*, $\mathbf{H} \neq \mathbf{C}$.

Although this is not a unique representation as far as Ψ_n is concerned (*vide* the discussion in Section V-A, for optimum systems), let us postulate that our receiver (except for a switch at $t = 0$, T) has a linear, time-invariant filter preceding the nonlinear operation. Then if $V_F(t)$ is the output of this filter and if $h_T(t - \tau)$ is its weighing function, so that

$$V_F(t) = \int_0^T V(t')h_T(t - t') \, dt', \quad (0 \leq t \leq T), \quad (131)$$

we can write the sampled data in matrix form, $\mathbf{V}_F = \mathbf{Q}\mathbf{V}$, where $\mathbf{Q} = [h_T(t_i - t_j)\Delta t]$; see Appendix V. Next, let us compute

$$\tfrac{1}{2}\psi_N^{-1}\tilde{\mathbf{V}}_F\mathbf{V}_F = \tfrac{1}{2}\psi_N^{-1} \sum_{jk}^n V_j V_k \sum_i^n h_T(t_i - t_j)\Delta t^2 \quad (132)$$

which suggests that we set

$$(\mathbf{D}_0)_{jk} = \sum_i^n h_T(t_i - t_j)h_T(t_i - t_k)(\Delta t)^2 \quad (133)$$

so that for band-limited white noise, sampled at the proper times $t_i = j/2B$ (in order that $\mathbf{k}_N = \mathbf{I}$), we have again

$$\Psi_n = \tilde{\mathbf{V}}\psi_N^{-1}\mathbf{D}_0\mathbf{V} = \tilde{\mathbf{V}}\mathbf{H}\mathbf{V} = \psi_N^{-1}\tilde{\mathbf{V}}_F\mathbf{V}_F. \quad (134)$$

By choosing h_T beforehand, we then specify $\mathbf{D}_0$ *a priori*, as mentioned above. Note that $\mathbf{Q}$ in effect diagonalizes $\mathbf{H}$ (see Appendix V) and in addition that the subsequent nonlinear operation has zero memory. In fact, the entire system here is identical with the one whose block diagram is shown in Fig. 2 except that now $\log L_n(\mathbf{V})$ is not in general optimum.

For continuous sampling in white noise backgrounds we find, as in Section V-A, that the structure factor is

$$\lim_{n\to\infty} \Psi_N \to \Psi_T = \frac{2}{W_{0N}} \int_0^T V_F(t)^2 \, dt$$

$$= \frac{2}{W_{0N}} \iint_0^T V(t_1)\rho_T(t_1, t_2)_0 V(t_2) \, dt_1 \, dt_2 \quad (135)$$

where

$$\rho_T(t_1, t_2)_0 = \rho_T(t_1 - t_2, t_1)_0$$

$$\equiv \int_0^T h_T(x - t_1)_0 h_T(x - t_2)_0 \, dx, \quad (0 \leq t_1, t_2 \leq T) \quad (136)$$

[28] Bibliography [4], Paper No. II for some explicit results in the case of the high-Q, LRC noise signal.
[29] Bibliography [1], (3) and (3.6).

as in (15), e.g., $(\mathbf{D}_0)_{jk} \rightarrow \rho_T(t_j, t_k)_0 dt$ in the limit $B \rightarrow \infty$ (or $n \rightarrow \infty$). The continuous analog of $\log L(\mathbf{V})$ is accordingly

$$\log L_T(V(t)) = G_0 + \frac{1}{W_{0N}} \int_0^T V_F(t)^2 \, dt$$

$$= G_0 + \frac{1}{W_{0N}} \iint_0^T V(t_1) \rho_T(t_1; t_2)_0 V(t_2) \, dt_1 \, dt_2. \quad (137)$$

Note that instead of a time-invariant matched filter and square-law rectifier, followed, as usual, by the ideal integrator $\int_0^T (\quad) dt$, (Fig. 2), we can equally well regard Ψ_T as computed by a linear, *time-varying* filter, and multiplier, as indicated in Fig. 1, following the argument of (74) and (75) above.

Although we have used an ideal integrator for the post-rectifier filter, since the system itself is not optimum we expect by using some other post-detection filter that an improvement in performance is possible. Returning to the discrete case for the moment, we can express this now as

$$\Psi_n' = \psi_N^{-1} \tilde{\mathbf{V}}_F' \mathbf{V}_F'$$

$$= \psi_N^{-1} \sum_i^n (V_F)_i^2 A_i, \quad \mathbf{V}_F' = [A_i (V_F)_i] \quad (138)$$

where the weighting A_i is normalized in some fashion: $A_i \equiv \gamma_T h_I(T - t_i)$, with $\gamma_T \equiv \int_0^T h_I(T - t) dt$, corresponding to $\sum_i^n A_i / n = 1$. The structure factor for continuous sampling becomes with the help of $\mathbf{V}_F = \mathbf{Q}\mathbf{V}$, etc. above,

$$\Psi_T' = \frac{2}{W_{0N}} \iint_0^T V(t_1) \rho_T'(t_1, t_2)_0 V(t_2) \, dt_1 \, dt_2 \quad (139)$$

where now

$$\rho_T'(t_1, t_2)_0 = \int_0^T \gamma_T h_I(T - x) h_T(x - t_1)_0 h_T(x - t_2) \, dx,$$

$$(0 \le t_1, t_2 \le T). \quad (139)$$

We still have the linear filter (h_T), followed by the zero-memory (full-wave) square law element, but with a post-rectifier filter that is no longer the simple ideal integrator of our previous discussion.

With the above in mind we can make several general statements about system performance:

1) *For the optimum receiver no improvement in performance is gained by using a post-detection filter other than the ideal integrator.* Any other filter will decrease the effectiveness of the system. This follows at once from the fact that the optimum system is Bayes and is unique.[30]

2) *When a nonoptimum receiver is used, post-detection filtering other than ideal integration can give improved performance.* (By improved performance is meant smaller average risk. This average risk, R, however, from the definition of the optimum system as one that

minimizes average risk, can never be less than the Bayes risk, R^*). By adjusting $h_I(T - x)$ it is possible to alter R, so that some h_I will lead to systems with smaller risk than when simple, ideal integration is employed ($h_I = T$). To prove this, we must calculate R under these various choices, but from physical considerations we see that post-detection filtering can be used to enhance the signal *vis-à-vis* the noise, if the predetection linear filter, h_T, is not optimum.

The above results are generalizations of earlier theorems found by Van Vleck and Middleton [8], who derived similar statements on the basis of calculations of signal-to-noise ratios at the output of a radar receiver. Here, however, the decision process is explicitly taken into consideration, and all information in the received wave as well as *a priori* data is used.

We note, finally, that to compare performances of ideal and nonideal systems, the error probabilities α, β must be obtained. These follow as in Section III, and for the structure (129) assumed here we get after some manipulation

$$\alpha = \int_{\log \mathcal{K}}^\infty Q_n(x)_0 \, dx; \quad \beta = \int_{-\infty}^{\log \mathcal{K}} P_n(x)_0 \, dx \quad (140)$$

where

$$Q_n(x)_0 = \int_{-\infty}^\infty \frac{e^{-i\xi(x - G_0)}}{[\det (\mathbf{I} - i\xi \mathbf{D}_0)]^{1/2}} \frac{d\xi}{2\pi};$$

$$P_n(x)_0 = \int_{-\infty}^\infty \frac{e^{-i\xi(x - G_0)}}{(\det [\mathbf{I} - i\xi \mathbf{D}_0(\mathbf{I} + a_0^2 \mathbf{D})])^{1/2}} \frac{d\xi}{2\pi} \quad (141)$$

with $a_0^2 = \psi_S / \psi_N$ once more.[31] The average risk is then computed from (3).

Exactly the same sort of relations are found when the signal is narrow-band and an alternative structure using the "components" V_C, V_S of the received wave is employed. We have, analogous to (127),

$$Q_n(y)_0 = \int_{-\infty}^\infty \frac{e^{-i\xi(y - G_0')}}{\det (\mathbf{I} - i\xi \mathbf{D}_0')} \frac{d\xi}{2\pi};$$

$$P_n(y)_0 = \int_{-\infty}^\infty \frac{e^{-i\xi(y - G_0')}}{\det [\mathbf{I} - i\xi \mathbf{D}_0'(\mathbf{I} + a_0^2 \mathbf{D})]} \frac{d\xi}{2\pi} \quad (142)$$

in which $\mathbf{G}_0'$, $\mathbf{D}_0'$ are the corresponding narrow-band versions of G_0 and $\mathbf{D}_0$. Trace methods enable us to evaluate α, β in the general case (141), as well as (142), for threshold reception, while the exact procedure using the eigenvalues of $\mathbf{D}_0'$, etc. may be successfully applied, as in Section VII, for all signal strengths when the signal is narrow-band.

IX. CONCLUSION

In the preceding sections we have formulated the problem of the optimum and suboptimum detection of normal noise signals in (additive) normal noise backgrounds, using the methods of statistical decision theory.

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[30] Bibliography [1], (2) and Appendix I.

[31] We assume that the noise is band-limited and white, $\psi_N = BW_{0N}$, and that sampling occurs at the intervals $t_j = j/2B$, so that $\mathbf{k}_N = \mathbf{I}$.

The general solutions for both broad- and narrow-band signal processes in either colored or white noise have been indicated and particular attention, along with a variety of specific results, has been given to the important class of systems that operate against white noise. While our main effort in this regard has been directed to weak-signal, or threshold operation (which is usually the more important situation in practice), the general structure of the optimum receiver has been found for all [(*a priori* known) rms] input signal levels and is represented in terms of physically realizable elements. The trace-method introduced here (Appendix II) enables us to obtain results in the threshold cases without having to calculate eigen-values explicitly and is convenient from the point of view of approximation when it is necessary to determine error probabilities and average risk for system operation.

It should be pointed out that the above detectors are optimum under the assumption of a known (rms) input signal level (as well as rms noise level). In many instances, however, this (rms) input signal level is not available to the observer, who at most has simply a distribution of such possible values. Then optimum performance must take this into account by suitable average in $\langle F(\mathbf{V} \mid \mathbf{S}) \rangle_s$, (5), (7), (8), and we may expect that the structure of the resulting system, including the bias, will be noticeably altered in many cases. Our present results can still be employed, if we *set* a_0^2, or ψ_s, at the level appropriate to a given value of minimum detectable signal; our system is optimum for all input signals of that particular strength, but is no longer optimum if the actual signal level is either above or below this value. This however, may not be serious, since above and below this value our receiver will still respond reasonably well: below this level we will reject the hypothesis "signal plus noise" anyway, and above this point we will accept it, albeit without the sensitivity of an optimum system.

Finally, we remark that the subject of the detection of random signals in noise is by no means exhausted here. Detailed solutions for colored noise backgrounds, the question of the unknown input signal level, strong-signal operation, and explicit calculations for suboptimum systems, including comparison with corresponding optimum receivers, all remain for later study. Extension to the case of fsk signals, and problems of more compli-cated signal waveforms (although some aspects of these questions are covered rather thoroughly in Price's work [4]) are also topics for further investigation.

Appendix I

Reduction of Det $(\mathbf{I} + \gamma \mathbf{G})$; Eigenvalue Method

A. *Preliminary Remarks*

One of our principal technical problems in obtaining useful expressions for the bias and the error probabilities associated with the detection systems discussed above is the reduction of quantities such as det $(\mathbf{I} + \gamma \mathbf{G})$ to a form more convenient for analysis and, ultimately, for compu-tation. In Appendix I we shall outline briefly one such

method of reduction and some of its byproducts, and reserve for Appendix II the discussion of an alternative approach.

Accordingly, we begin by assuming that:
1) γ is in general a complex quantity;
2) $\mathbf{I}$ is the unit matrix; and
3) $\mathbf{G}$ is an $(\mathbf{n} \times \mathbf{n})$ matrix all of whose elements are real quantities. For the moment we shall *not* require that $\mathbf{G}$ be symmetrical. From assumption 3) we can always find an $(\mathbf{n} \times \mathbf{n})$ matrix $\mathbf{Q}$ which diagonalizes $\mathbf{G}$ by the similarity transformation[32]

$$\mathbf{Q}^{-1}\mathbf{G}\mathbf{Q} = \mathbf{\Lambda} = [\lambda_j \delta_{jk}] \quad \text{or} \quad \mathbf{G}\mathbf{Q} = \mathbf{\Lambda}\mathbf{Q} = \mathbf{Q}\mathbf{\Lambda}, \quad (143)$$

where $\lambda_j (j = 1, \cdots, n)$ are the n distinct eigenvalues of $\mathbf{G}$ and δ_{jk} is the familiar Kronecker delta $\delta_{jk} = 0$, $(j \neq k)$; $\delta_{jj} = 1$. Thus, if we let the (column) vectors, $\mathbf{f}_j = [f_{(\,)j}]$ be the eigenvectors, corresponding to the eigenvalues λ_j in (143) above, the diagonalizing matrix $\mathbf{Q}$ is then simply the $(\mathbf{n} \times \mathbf{n})$ matrix formed by the $\mathbf{f}_j$ as columns,[33] *e.g.*, $\mathbf{Q} = [\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_j, \cdots, \mathbf{f}_n]$. Taking the lth row and jth column of $\mathbf{G}\mathbf{Q} = \mathbf{\Lambda}\mathbf{Q}$, (143) above, we may write

$$(\mathbf{G}\mathbf{Q})_{lj} = \sum_{k=1}^{n} G_{lk}Q_{kj} = \lambda_j Q_{lj}, \quad (144a)$$

or since $\mathbf{Q}_{lj} = f_{lj}$ we get for the lth row of (143)

$$\sum_{k}^{n} G_{lk}f_{kj} = \lambda_j f_{lj}, \quad (l = 1, \cdots, n), \quad (j = 1, \cdots n), \quad (144b)$$

or in vector form

$$\mathbf{G}\mathbf{f}_j = \mathbf{\Lambda}\mathbf{f}_j, \quad (j = 1, \cdots, n). \quad (144c)$$

The eigenvalues are first found by solving the secular equation det $(\mathbf{G} - \lambda\mathbf{I}) = 0$, while the eigenvectors are then obtained from (144b), except for an arbitrary constant in each component. When $\mathbf{G}$ is symmetrical, $\mathbf{Q}$ can be made an orthogonal transformation, *e.g.*, $\tilde{\mathbf{Q}}\mathbf{Q} = \mathbf{I}$, and the arbitrary constants are removed. The ortho-normalizing condition for the n eigenvectors is then

$$\sum_{k=1}^{n} Q_{kl}Q_{kj} = \delta_{lj}, \quad \text{or} \quad \sum_{k=1}^{n} f_{kl}f_{kj} = \delta_{lj}, \quad (145)$$

while the eigenvalues are found as before from det $(\mathbf{G} - \lambda\mathbf{I}) = 0$, and the eigenvectors from the n linearly independent relations (144b), subject now to (145). For symmetrical or unsymmetrical matrices [satisfying (3)], in any case, we can use (143) and the fact that det $(\mathbf{AB}) = $ det $\mathbf{A}$ det $\mathbf{B}$ to write[34]

det $(\mathbf{I} + \gamma\mathbf{G}) = $ det $(\mathbf{Q}^{-1}\mathbf{Q} + \gamma\mathbf{Q}^{-1}\mathbf{G}\mathbf{Q})$

$$= \text{det } (\mathbf{I} + \gamma\mathbf{\Lambda}) = \prod_{j=1}^{n} (1 + \gamma\lambda_j). \quad (146)$$

For the case of a colored noise "signal" in a white noise

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[32] See, for example, Bibliography [12], (10.15a). Also (10.14) and (10.9c).

[33] See Bibliography [12], ch. 10.

[34] Even when the eigenvalues of $\mathbf{G}$ are not distinct and $\mathbf{G}$ is unsymmetrical, it is still possible to write $\Pi^{n}_{j=1}, (1 + \gamma\lambda_j)$ for det $(\mathbf{I} + \gamma\mathbf{G})$, but there no longer exists a matrix $\mathbf{Q}$, and hence a set of eigenvectors $\mathbf{f}_j$, which can be used to diagonalize $\mathbf{G}$, see Biblio-graphy [12], (10-15b), as $\mathbf{G}$ cannot then be put into completely diagonal form.

background, $\mathbf{G}$ is proportional to $\mathbf{k}_S$, the covariance matrix of the signal, which is symmetrical; $\mathbf{Q}$ can therefore be an orthogonal matrix, and we can relax the requirement above on $\mathbf{G}$ that all its eigenvalues be distinct. On the other hand, for the more general problem of colored noise in colored noise $\mathbf{G}$ is proportional to $\mathbf{k}_S \mathbf{k}_N^{-1}$, and although both $\mathbf{k}_S$ and $\mathbf{k}_N$ are symmetrical, their product is not (except in the unlikely situation that $\mathbf{k}_S$ and $\mathbf{k}_N$ commute). To diagonalize $\mathbf{G}$ as in (143) above, we must reimpose the condition that its eigenvalues be distinct. For the physical processes discussed here, this is assumed always to be the case.

B. Continuous Sampling

In most of our present applications continuous sampling is ultimately postulated. Thus, in going from the discrete to the continuous process the intervals between sampled values are allowed to become arbitrarily close, while the total number, n, of sampled values becomes infinite. Two situations are distinguished: 1), where the observation interval $(0, T)$ remains finite; and 2), where $T \to \infty$; in either case it is assumed that $\lim_{n \to \infty} \mathbf{G}$ goes over into $G(t_1, t_2)$ all t_1, t_2 in $(0, T)$, where G has suitable continuity and convergence properties. Our determinant (146) becomes

$$\mathfrak{D}_T(\gamma) \equiv \lim_{n \to \infty} \det(\mathbf{I} + \gamma \mathbf{G}) = \prod_{j=1}^{\infty} (1 + \gamma \lambda_j^{(T)}); \quad (147)$$

$\mathfrak{D}_T(\gamma)$ is known as the Fredholm determinant [12] and since the eigenvalues of G are all distinct, we may have recourse to the usual Hilbert theory to write $\lambda_j^{(T)}$ as the limiting form of the eigenvalues λ_j when $n \to \infty$, $(T < \infty)$ [see (149) *et seq.*]. The determinant $\mathfrak{D}_T(\gamma)$ is absolutely convergent [13] for all $(0 \le (t_1, t_2) \le T)$ provided $|\gamma G(t_1, t_2)| \le M_G$, where M_G is the maximum value of $|\gamma G|$ in the region $0 \le (t_1, t_2) \le T$. The same condition still applies in the second situation of infinite observation periods $(T \to \infty)$, where the region of convergence is now $(0 \le t_1, t_2 < \infty)$, and where $\mathfrak{D}_T(\gamma) \to \mathfrak{D}_\infty(\gamma)$.

We shall need some further properties of the eigenvalues λ_j, $\lambda_j^{(T)}$, etc. Returning for the moment to the discrete case $(n < \infty)$ again, we may write the well-known result for any matrix $\mathbf{G}$ (here with distinct eigenvalues):

$$\sum_{j=1}^{n} \lambda_j^m = \text{trace } \mathbf{G}^m, \quad (m \ge 0). \quad (148)$$

To obtain limiting forms as $n \to \infty$, let us first require that $T < \infty$ and then set $t_k = kT/n$, $\Delta t = T/n$, etc. We assume that $\lim_{n \to \infty} T/n \lambda_j = \lambda_j^{(T)}$ exists (all j), with $\lambda_j^{(T)}$ discrete, (an assumption here that follows from the Hilbert theory of integral equations). Multiplying both sides of (148) by $(T/n)^m$ and passing to the limit then gives

$$\sum_{j=1}^{\infty} [\lambda_j^{(T)}]^m = \lim_{n} \left\{ \left(\frac{T}{n}\right)^m \sum_{l_1 \cdots l_m} G_{l_1 l_2} G_{l_2 l_3} \cdots G_{l_m l_1} \right\}$$

$$= \int \cdots \int_0^T G(t_1, t_2) G(t_2, t_3) \cdots G(t_m, t_1) \, dt_1 \cdots dt_m$$

$$\equiv B_m^{(T)}, \quad (m \ge 1). \quad (149a)$$

For $T \to \infty$ we get in similar fashion

$$\sum_{j=1}^{\infty} [\lambda_j^{(\infty)}]^m = \int \cdots \int_0^\infty G(t_1, t_2) \cdots G(t_m, t_1) \, dt_1 \cdots dt_m$$

$$\equiv B_m^{(\infty)}, \quad (m \ge 1). \quad (149b)$$

In this last instance G must be such that the integrals exist, or equivalently that the series converge. The $B_m^{(T)}$, $B_m^{(\infty)}$ are known as the *iterated kernels* of a certain class of integral equations, described below briefly.

Another useful expression for $\det(\mathbf{I} + \gamma \mathbf{G})$, which may be carried over to the continuous case with the help of the above, is given by the expansion of the determinant as a polynomial in γ:[35]

$$\det(\mathbf{I} + \gamma \mathbf{G}) = \sum_{m=0}^{n} \frac{\gamma^m}{m!} D_m^{(n)};$$

$$D_m^{(n)} = \sum_{l_1 \cdots l_m} \begin{bmatrix} G_{l_1 l_1} & G_{l_1 l_2} \cdots G_{l_1 l_m} \\ G_{l_2 l_1} & G_{l_2 l_2} & \vdots \\ \vdots & & \ddots \\ \vdots & & \ddots \\ G_{l_m l_1} \cdots \cdots \cdots \cdots G_{l_m l_m} \end{bmatrix}$$

$$(1 \le m \le n). \quad (150)$$

Evaluating the determinants $D_m^{(n)}$ shows that we can write $D_m^{(n)}$ as a function of the traces of $\mathbf{G}$, $\mathbf{G}^2 \cdots$, $\mathbf{G}^m$, *viz.*:

$$D_m^{(n)} = D_m^{(n)} (\text{trace } \mathbf{G}, \text{trace } \mathbf{G}^2, \cdots, \text{trace } \mathbf{G}^m). \quad (151)$$

The corresponding limiting form for continuous sampling $(n \to \infty)$ is given in (161c) following.

The integral equations from which the $\lambda_j^{(T)}$, $\lambda_j^{(\infty)}$ are found can be obtained directly from (144b). The matrix $\mathbf{G}$ goes over into $G(t_1, t_2)$, the eigenvectors $\mathbf{f}_i$ become the eigenfunctions f_i, and instead of the sum in (144b) one gets an integral. Formally, let us multiply both sides of (144b) by T/n, with $t_k = kT/n$, $\Delta t = T/n$, etc. as before, and pass to the limit $(n \to \infty)$, making the assumption above that $\lim_{n \to \infty} (T/n)\lambda_i = \lambda_i^{(T)}$ as justified by the Hilbert theory, if the $\lambda_i^{(T)}$ (and λ_i) are discrete and distinct (and if there are at most a finite number of (finite) negative eigenvalues $\lambda_i^{(T)}$). The result is

$$\int_0^T G(t, \tau) f_i(\tau)_T \, d\tau = \lambda_i^{(T)} f_i(t)_T; \quad (0 \le t \le T);$$

$$(j = 1, 2, \cdots, \infty) \quad (152a)$$

$$\int_0^\infty G(t, \tau) f_i(\tau) \, d\tau = \lambda_i^{(\infty)} f_i(t), \quad (0 \le t \le \infty). \quad (152b)$$

For discreteness of the eigenvalues we require that $G(t, \tau)$ be quadratically integrable[36] in the respective regions $(0 \le t, \le T)$, $(0 \le t, \le \infty)$. If $\mathbf{G}$ is symmetrical, or equivalently if $G(t_1, t_2) = G(t_2, t_1)$ in the continuous case, it is a straightforward matter to insure that the eigenfunctions belong to a complete orthonormal set; the condition for this follows directly from (145) and is

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[35] See Bibliography [13], (15), p. 24.
[36] See, for example, Bibliography [14], the footnote preceding (4.10b).

$$\int_0^{T,(\infty)} f_i(t)f_k(t)\, dt = \delta_{ik} \qquad (152c)$$

where the f_j, f_k are, of course, solutions of (152b) if $(T \to \infty)$. Notice, however, that the integrals (152) apply in any case, even if $G(t_1, t_2)$ does not equal $G(t_2, t_1)$, as long as the eigenvalues are discrete.

Eigenvalues and eigenfunctions for a number of different kernels G are listed in the table below. Although we consider only Cases I and II specifically in the present paper, we include some results for more involved situations as well, along with a brief description of the type of problem in which they may occur. Details of the solutions are available in the references. The kernels here take the general form $G(t_1, t_2) = A(t_1)K(|t_1 - t_2|)$, where $\int_{-\infty}^{\infty} A(t)dt$ exists as indicated in Table II.

For case I in Table II, approximate expressions for eigenvalues of high order are readily obtained. To a first approximation we may write

$$\text{Case I:} \quad \lambda_i \doteq \frac{2\kappa a^2 T g}{\pi^2(j-1)^2 + g(4+g)} \; ;$$

$$j \gg 1 \qquad g \equiv cT. \qquad (153)$$

Higher order corrections to these results may be found by successive approximations. We note for all the cases listed in Table II that the eigenvalues are positive (since A,

$\kappa > 0$) and approach zero 0 (j^2) as $j \to \infty$. The various iterated kernels, $B_m^{(T)}$, $B_m^{(\infty)}$, (149a) and (149b), are therefore finite, since the series in $\lambda_j^{(T)m}$, etc. are clearly convergent ($m \geq 1$). In some instances, we may have negative eigenvalues, as well, if $A(u)$ is less than zero for some ranges of u, but at most these will be limited to a finite number of (discrete) eigenvalues.[37] We remark, finally, that for certain choices of the parameter $\nu = c/b$, in Case III, *viz.*, $\nu = 1/2, 3/2$, the eigenvalues can be given precisely for all orders:

$$\left.\begin{array}{l}
\text{Case III:} \quad \nu = 1/2: \\[2mm]
J_{-1/2}(q_i) = \sqrt{\dfrac{2}{\pi q_i}} \cos q_i = 0; \\[4mm]
\therefore \lambda_j = \dfrac{8c\kappa a^2}{(2j-1)^2 \pi^2 b^2} \\[4mm]
\nu = 3/2: \\[2mm]
J_{1/2}(q_i) = \sqrt{\dfrac{2}{\pi q_i}} \sin q_i = 0; \\[4mm]
\therefore \lambda_j = \dfrac{2c\kappa a^2}{j^2 \pi^2 b^2}
\end{array}\right\} \quad j \geq 1 \qquad (154)$$

[37] See, for example, Bibliography [14], the discussion following (4.15b), where $A(u)$ may represent the weighting function of a CR-video filter, *e.g.*, $A(u) = \delta(u - 0) - (RC)^{-1} e^{-u/RC}$, $u > 0-$.

TABLE II
THE INTEGRAL EQUATION

$$\int_0^T A(u)K(|t - u|)f_i(u)\, du = \lambda_i^{(T)}f_i(t), \qquad (0 \leq t \leq T)$$

Model [a^2, b, η, $K \geq 0$]	$A(u)$	$K(|t-u|)$	Eigenvalues	Eigenfunctions
I) [$b = 0$; $T < \infty$]; distribution of $\int_0^T x(t)^2 dt$; $x(t)$ a Gauss process with zero mean; 1) $x = $ RC noise, or 2) $x = $ high-Q, LRC noise [5].	a^2	$\kappa e^{-c|t-u|}$	$\lambda_i = \dfrac{2\kappa a^2}{c(1 + q_i^2)}$; $\tan(cTq)_i = \dfrac{-2q_i}{1 - q_i^2}$ all $q_i > 0$, *i.e.*, $\lambda_i > 0$)	$A_i e^{-i\gamma_i t} + B_i\ e^{i\gamma_i t}$; or $A_i' \cos \gamma_i(t - T/2)$ $+ B_i' \sin \gamma_i(t - T/2)$ $\left\{\begin{array}{l}\gamma_i = cq_i \\ = c\sqrt{2\kappa a^2/c\lambda_i - 1}\end{array}\right\}$
II) ($0 < T < \infty$); 1) Markoff scatter; $A(u) = (ae^{-bu})^2$, optimum detection of RC or high-Q, LRC noise "signal" in white noise [4]. 2) RC or high-Q, LRC noise into a square-law detector, followed by an RC video filter, $A(u)$. Distribution of $\int_0^T A(t-u)$ $I(u)^2 du$.	$a^2 e^{-2bu}$	$\kappa e^{-c|t-u|}$	q_i (positive) roots of $J_{\nu+1}(e^{-bT}q_i)N_{\nu-1}(q_i) =$ $N_{\nu+1}(e^{-bT}q_i)J_{\nu-1}(q_i);$ $q_i = \sqrt{\dfrac{2c\kappa a^2}{\lambda_i b^2}}$ $\nu = c/b$	$B_i\left\{\dfrac{-N_{\nu-1}(q_i)}{J_{\nu-1}(q_i)} \cdot J_\nu(q_i e^{-bt})\right.$ $\left. + N_\nu(q_i e^{-bt})\right\}$
III) ($T \to \infty$); 1) As in 1), II above. 2) As in 2), II above; Juncosa's integral equation [15]. See also Kac and Siegert [5].	$a^2 e^{-2bu}$	$\kappa e^{-c|t-u|}$	q_i (positive) roots of $J_{\nu-1}(q_i) = 0;$ $q_i = \sqrt{\dfrac{2c\kappa a^2}{\lambda_i b^2}}$ $\nu = c/b$	$A_i J_\nu\left(\sqrt{\dfrac{2c\kappa a^2}{b^2\lambda_i}}\ e^{-bt}\right)$

and these results may then be used to sum the various series for the bias, error probabilities, and average risk, in these special cases.[38] The similar nature of these integral equations stems from the fact that the underlying probability mechanisms are all normal, although the problems in which they arise are physically quite different.

Appendix II
Reduction of Det $(I + \gamma G)$; Trace Method; Evaluation of Integrals

A. Fundamental Identity

With the $(n \times n)$ matrix G of Appendix I-A, let us examine the determinantal expansion (150) of det $(I + \gamma G)$ in more detail. From this, in fact, we can establish the following identity, which is basic to the so-called trace method for reducing expressions like det $(I + \gamma G)$ to more manageable forms and which is particularly suited in detection (and extraction) theory to problems of threshold performance.[39] The identity in question is specifically

$$\exp\left\{-\sum_{m=1}^{\infty} \frac{(-1)^m}{m} \gamma^m \text{ trace } G^m\right\} \equiv \det (I + \gamma G), \quad (155)$$

which holds whenever the exponential series converges. Proofs of (155) can be given in several ways. A direct method, based on the determinantal expansion (150), is to develop both sides of (155) in a power series, compare coefficients of γ^k, and observe for all $k > n$ that the coefficients of γ^k are identically zero. For $k \leq n$ we get simply the determinantal expansion (150) in both members of (155), with $D_k^{(n)}$ the coefficient of γ^k. Thus, from (150) we have (whenever the m series is convergent)

$$\exp\left\{\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \gamma^m \text{ trace } G^m\right\} = \sum_{k=0}^{n} \frac{\gamma^k}{k!} D_k^{(n)}, \quad (156)$$

and the coefficients $D_k^{(n)}$ are specifically

$$D_0^{(n)} = 1; \quad D_1^{(n)} = \text{trace } G;$$
$$D_2^{(n)} = \text{trace}^2 G - \text{trace } G^2; \quad (157)$$

$D_3^{(n)} = \text{trace}^3 G - 3 \text{ trace } G \text{ trace } G^2 + 2 \text{ trace } G^3;$
$D_4^{(n)} = \text{trace}^4 G - 6 \text{ trace}^2 G \text{ trace } G^2 + 3 \text{ trace}^2 G^2$
$\qquad + 8 \text{ trace } G \text{ trace } G^3 - 6 \text{ trace } G^4, \text{ etc;}$

with the higher-order terms obtainable from (150), or from the expansion of the left member of the identity (155).

This direct approach, however, does not readily reveal the conditions of convergence for the exponential series. A more elegant demonstration of (155), which at the same time establishes the desired interval of convergence, starts with the relations (146) and (148), *viz.*,

$$\det (I + \gamma G) = \prod_{j=1}^{n} (1 + \gamma\lambda_j);$$

$$\sum_{j=1}^{n} \lambda_j^m = \text{trace } G^m, \quad (m \geq 1), \quad (158)$$

[38] See Bibliography [5], (7.24)-(7.29), for example.
[39] To the author's knowledge, the particular relation (155) does not appear to have been noted previously in problems of this type.

and writes

$$\det (I + \gamma G) = \exp\left\{\log \prod_{j=1}^{n} (1 + \gamma\lambda_j)\right\}$$
$$= \exp\left\{\sum_{j=1}^{n} \log (1 + \gamma\lambda_j)\right\}. \quad (159)$$

Now let the n distinct eigenvalues λ_j $(j = 1, \cdots, n)$ be arranged in descending order of magnitude, with $|\lambda_1|$ the magnitude of the largest, *e.g.*, $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_j| > \cdots |\lambda_n|$. Then the logarithm can be expanded in an absolutely convergent series for all λ_j, provided $|\gamma\lambda_1| < 1$, to give

$$\sum_{j=1}^{n} \log (1 + \gamma\lambda_j) = \sum_{j=1}^{n} \sum_{m=1}^{\infty} (-1)^{m-1}(\gamma\lambda_j)^m/m$$
$$= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \gamma^m\left(\sum_{j=1}^{n} \lambda_j^m\right), \quad |\gamma\lambda_1| < 1, \quad (160)$$

where the interchange of series is permitted, since the m series is absolutely convergent $|\gamma\lambda_1| < 1$. But the series over j is just trace G^m, (158), and so the identity (155) is established. The region of convergence in the complex γ plane is determined solely by the largest eigenvalue of G and is a circle of radius $r < |\lambda_1|^{-1}$. For example, in the calculation of the bias term, (25), γ is equal to a_0^2, so that the left member of (155) is an exact expression for det $(I + a_0^2 G)$ whenever $a_0^2 < |\lambda_1|^{-1}$; det $(I + a_0^2 G)$, as given by (150), is, of course, defined for all $a_0^2 (< \infty)$. The convergence condition suggests, then, that our exponential representation should be particularly useful in threshold situations, where a_0^2 is $0(|\lambda_1|)$ or smaller. [Note, incidentally, that starting with the relation $\sum_j^n \lambda_j^m = \text{trace } G^m$, we can also establish the expansion (150), or given (155), we can in a similar way prove that (148) is valid.]

In the limit $(n \to \infty)$ of continuous sampling, trace G^m is replaced by the iterated kernel $B_m^{(T)}$ in the identity (155), while $\lambda_j \to \lambda_j^{(T)}$, and det $(I + \gamma G)$ becomes the Fredholm determinant $\mathfrak{D}_T(\gamma)$; see (148), (149). To see this, we simply repeat the steps (158)-(160), where

$$\lim_{n\to\infty} \sum_{j=1}^{n} \left(\frac{T}{n}\right)^m \lambda_j^{(T)m} = \lim_{n\to\infty} \left(\frac{T}{n}\right)^m \text{trace } G^m = B_m^{(T)}, \quad (161a)$$

in the manner of (149a). The basic identity (155) is now specifically

$$\mathfrak{D}_T(\gamma) = \prod_{j=1}^{\infty} (1 + \gamma\lambda_j^{(T)}) = \exp\left\{\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \gamma^m B_m^{(T)}\right\},$$
$$\text{all } |\gamma| < |\lambda_1^{(T)}|^{-1}, \quad (161b)$$

with similar results for semi-infinite observation periods $(T \to \infty)$. From this the continuous analog of (151) follows at once on expanding the last member of (149b) as a power series in γ, namely

$$\mathfrak{D}_T(\gamma) = \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} D_m^{(\infty)}(B_1^{(T)}, \cdots, B_m^{(T)}). \quad (161c)$$

Observe that this relation and the first equation of (161b) are in fact valid for all γ, since the Fredholm determinant

absolutely and permanently convergent for the kernels assumed here [13].

Evaluation of Integrals: Characteristic Function and Distribution Densities

Let us now use both the fundamental identity (155) and the eigenvalue form of det $(\mathbf{I} + \gamma\mathbf{G})$ to evaluate a class of integrals that arises in detection and extraction problems whenever normal noise is present. It is convenient to consider this as a problem in probability distributions. We write first

$$x = \log \Lambda_n(\mathbf{z}), \qquad (162a)$$

here specifically the transformation between $\mathbf{z}$ and x is

$$x = \log \Lambda_n(\mathbf{z}) = C_0 + \tilde{\mathbf{C}}_1\mathbf{z} + \tfrac{1}{2}\tilde{\mathbf{z}}\mathbf{C}_2\mathbf{z}, \qquad (162b)$$

which C_0 is a scalar, $\mathbf{C}$ is a column vector, like $\mathbf{z}$, of n rows, and $\mathbf{C}_2$ is a (real), symmetrical $(n \times n)$ matrix. We ask now for the distribution density $W_1(x)$ of the random variable x, when $\mathbf{z}$ is normal, with mean values $[\bar{z}]$ and covariances $[(z_j - \bar{z}_j)(z_k - \bar{z}_k)]$, e.g., when the distribution density of z is

$$W_n(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \mathbf{F}}}$$

$$\cdot \exp \left\{ -\tfrac{1}{2}(\tilde{\mathbf{z}} - \tilde{\mathbf{z}})\mathbf{F}^{-1}(\mathbf{z} - \bar{\mathbf{z}}) \right\} \qquad (163)$$

$$\mathbf{F} = [(z_j - \bar{z}_j)(z_k - \bar{z}_k)] = [F_{jk}] = \tilde{\mathbf{F}};$$

he (˜) indicates the transposed matrix].

The probability density $W_1(x)$ can be expressed in terms of its characteristic function $F_1(i\xi)_x$ as

$$W_1(x) = \int_{-\infty}^{\infty} F_1(i\xi)_x e^{-i\xi x}\, \frac{d\xi}{2\pi},$$

$$\text{with} \quad F_1(i\xi)_x = \int_{-\infty}^{\infty} W_1(x)e^{i\xi x}\, dx. \qquad (164)$$

sing the transformation (162a) we can alternatively rite $W_1(x)$ as

$$W_1(x) = \int_{(\mathbf{z})} W_n(\mathbf{z})\, \delta(x - \log \Lambda_n(\mathbf{z}))\, d\mathbf{z}. \qquad (165)$$

ith the help of the representation

$$x - \log \Lambda_n(\mathbf{z}))$$

$$= \int_{-\infty}^{\infty} \exp \left\{ -i\xi(x - \log \Lambda_n(\mathbf{z})) \right\} \frac{d\xi}{2\pi} \qquad (166)$$

e find that the characteristic function of x becomes

$$(i\xi)_x \equiv \overline{e^{i\xi x}} = \langle \exp \left\{ i\xi \log \Lambda_n(\mathbf{z}) \right\} \rangle_z$$

$$= \int_{(\mathbf{z})} \exp \left\{ i\xi \log \Lambda_n(\mathbf{z}) \right\} W_n(\mathbf{z})\, d\mathbf{z}$$

$$= \exp \left\{ i\xi C_0 - \tfrac{1}{2}\tilde{\mathbf{z}}\mathbf{F}^{-1}\bar{\mathbf{z}} \right\}$$

$$\cdot \int_{-\infty}^{\infty} \cdots \int \frac{\exp\{i\tilde{\mathbf{z}}(\xi\mathbf{C}_1 - i\mathbf{F}^{-1}\bar{\mathbf{z}})\}\exp\{-\tfrac{1}{2}\tilde{\mathbf{z}}(\mathbf{F}^{-1} - i\xi\mathbf{C}_2)\mathbf{z}\}\, d\mathbf{z}}{(2\pi)^{n/2}\sqrt{\det \mathbf{F}}},$$

this last from (162b) and (163). Since $\det \mathbf{A} = (\det \mathbf{A}^{-1})^{-1}$, we use the relation[40] that

$$\int_{-\infty}^{\infty} \cdots \int \exp \left\{ i\tilde{\mathbf{z}}\mathbf{u} - \tfrac{1}{2}\tilde{\mathbf{z}}\mathbf{A}\mathbf{z} \right\} d\mathbf{z}$$

$$= (2\pi)^{n/2}(\det \mathbf{A})^{-1/2} \exp \left\{ -\tfrac{1}{2}\tilde{\mathbf{u}}A^{-1}\mathbf{u} \right\} \qquad (167)$$

to obtain finally

$$F_1(i\xi)_x = \frac{\exp \left\{ -\tfrac{1}{2}\tilde{\mathbf{z}}\mathbf{F}^{-1}(\mathbf{I} - \mathbf{\Delta}^{-1}(i\xi))\bar{\mathbf{z}} + i\xi[C_0 + \tfrac{1}{2}\tilde{\mathbf{z}}\{\mathbf{\Delta}^{-1}(i\xi) + \mathbf{F}^{-1}\mathbf{\Delta}^{-1}(i\xi)\mathbf{F}\}\mathbf{C}_1] - \tfrac{1}{2}\xi^2\tilde{\mathbf{C}}_1\mathbf{\Delta}^{-1}(i\xi)\mathbf{F}\mathbf{C}_1 \right\}}{\sqrt{\det \mathbf{\Delta}(i\xi)}} \qquad (168a)$$

where

$$\mathbf{\Delta}(i\xi) = \mathbf{I} - i\xi\mathbf{F}\mathbf{C}_2. \qquad (168b)$$

Our next step is to put the exponent of (168a) into a form more convenient for manipulation. We let $\mathbf{G} = \mathbf{F}\mathbf{C}_2$ and then diagonalize $\mathbf{G}$ with the similarity transformation $\mathbf{Q}$; see (143). We note, then, that

$$\mathbf{\Delta}(i\xi)^{-1} = \mathbf{Q}\{\mathbf{Q}^{-1}\mathbf{\Delta}(i\xi)^{-1}\mathbf{Q}\}\mathbf{Q}^{-1} = \mathbf{Q}[\mathbf{I} - i\xi\mathbf{Q}^{-1}\mathbf{G}\mathbf{Q}]^{-1}\mathbf{Q}$$

$$= \mathbf{Q}(\mathbf{I} - \mathbf{\Lambda})^{-1}\mathbf{Q}^{-1}; \qquad \mathbf{\Lambda} = [i\xi\lambda_j\delta_{jk}]. \qquad (169)$$

Applying this to the various quadratic forms in (168a), we easily find that

$$\tilde{\mathbf{z}}\mathbf{F}^{-1}(\mathbf{I} - \mathbf{\Delta})^{-1}\bar{\mathbf{z}} = -\tilde{\mathbf{z}}\mathbf{F}^{-1}\mathbf{\Gamma}_1\bar{\mathbf{z}}; \qquad \mathbf{\Gamma}_1 \equiv \left[\frac{i\xi\lambda_j}{1 - i\xi\lambda_j}\, \delta_{jk} \right]$$

$$\tilde{\mathbf{z}}\mathbf{\Delta}^{-1}\mathbf{C}_1 = \tilde{\mathbf{z}}\mathbf{\Gamma}_2\mathbf{C}_1; \qquad \mathbf{\Gamma}_2 \equiv \left[\frac{\delta_{jk}}{1 - i\xi\lambda_j} \right] \qquad (170)$$

$$\tilde{\mathbf{z}}\mathbf{F}^{-1}\mathbf{\Delta}^{-1}\mathbf{F}\mathbf{C}_1 = \tilde{\mathbf{z}}(\mathbf{I} - \mathbf{\Lambda})^{-1}\mathbf{C}_1 = \tilde{\mathbf{z}}\mathbf{\Gamma}_2\mathbf{C}_1$$

$$\tilde{\mathbf{C}}_1\mathbf{\Delta}^{-1}\mathbf{F}\mathbf{C}_1 = \tilde{\mathbf{C}}_1\mathbf{F}\mathbf{\Gamma}_2\mathbf{C}_1.$$

Letting

$$\bar{z}_j \cdot \sum_k^n z_k(\mathbf{F}^{-1})_{jk} \equiv a_j; \qquad \bar{z}_j \cdot C_j \equiv b_j;$$

$$C_j \sum_k^n C_k F_{jk} \equiv c_j, \qquad (171)$$

we can write the characteristic function (168a) alternatively as

$$F_1(i\xi)_x \qquad (172)$$

$$= e^{i\xi C_0} \prod_{j=1}^{n} \left\{ \frac{\exp \left\{ i\xi(b_j + \tfrac{1}{2}\lambda_j a_j) - \tfrac{1}{2}\xi^2 c_j \right\}/(1 - i\xi\lambda_j)}{(1 - i\xi\lambda_j)^{1/2}} \right\},$$

where we have used (146) for $\det \mathbf{\Delta}(i\xi)$, with $\gamma = -i\xi$.

The probability density of x, (164), cannot be given in closed form, generally, even if $a_j = b_j = c_j = 0$ (all j). With the help of the identity (155) we can, however, obtain an asymptotic approximation which is useful in

[40] See Bibliography [7], (11) and (12).

all cases of threshold performance. To do this we first expand the exponent of (172), retaining terms $0(\xi, \xi^2)$ only, and then develop the rest in a series in ξ. For example, the exponent of (172) becomes

$$\sum_j^n \left\{ \frac{i\xi(b_j + \frac{1}{2}\lambda_j a_j) - \frac{1}{2}\xi^2 c_j}{1 - i\xi\lambda_j} \right\}$$

$$= E_1^{(n)} \cdot i\xi - \frac{\xi^2}{2} E_2^{(n)} + 0(i^3\xi^3), \qquad (173)$$

where

$$E_1^{(n)} \equiv \sum_{j=1}^n \left(b_j + \lambda_j \frac{a_j}{2} \right);$$

$$E_2^{(n)} \equiv \sum_{j=1}^n \left[c_j + 2\lambda_j \left(b_j + \lambda_j \frac{a_j}{2} \right) \right], \quad \text{etc.,} \qquad (173a)$$

and higher terms are readily computed in similar fashion. From (155) we have

$$\prod_{j=1}^n (1 - i\xi\lambda_j)^{-1/2}$$

$$= \exp \left\{ \sum_{m=1}^\infty \frac{(-1)^m}{2m} (-i\xi)^m \text{ trace } \mathbf{G}^m \right\},$$

$$= \exp \left\{ \frac{i\xi}{2} \text{ trace } \mathbf{G} - \frac{\xi^2}{4} \text{ trace } \mathbf{G}^2 + 0(i^3\xi^3) \right\}, \qquad (174)$$

so that the characteristic function is now

$$F_1(i\xi)_x \doteq e^{i\xi(C_0 + E_1^{(n)} + \frac{1}{2} \text{ trace } \mathbf{G})}$$

$$\cdot \exp \left\{ -\frac{\xi^2}{2} (E_2^{(n)} + \frac{1}{2} \text{ trace } \mathbf{G}^2) \right\} \{1 + 0(i\xi)^3)\}. \qquad (175)$$

The corresponding probability density is asymptotically normal, with mean and variance

$$\bar{x} = C_0 + E_1^{(n)} + \frac{1}{2} \text{ trace } \mathbf{G};$$

$$\overline{x^2} - \bar{x}^2 \equiv \sigma_x^2 = E_2^{(n)} + \frac{1}{2} \text{ trace } \mathbf{G}^2, \qquad (176)$$

viz.,

$$W_1(x) = \frac{\exp \{-(x - \bar{x})^2/2\sigma_x^2\}}{\sqrt{2\pi\sigma_x^2}}. \qquad (177)$$

Correction terms, revealing the approach to the normal law, are found in straightforward fashion from the Fourier transform of the terms $0((i\xi)^3)$, and higher, in the characteristic function (175) above. The method is illustrated presently in a number of special cases [see (177) *et seq.*] and is recognized as one form of the method of steepest descents.

Although we cannot obtain $W_1(x)$ from (172) in closed form, it is possible to calculate any desired moment of x exactly, with the help of

$$\overline{x^m} = (-i)^m \frac{d^m}{d\xi^m} F_1(i\xi)_x \Big|_{\xi=0}. \qquad (178)$$

The first two moments here are found to be

$$\bar{x} = C_0 + E_1^{(n)} + \frac{1}{2} \text{ trace } \mathbf{G}, \qquad (\mathbf{G} = \mathbf{FC}_2), \qquad (179a)$$

$$\overline{x^2} = E_2^{(n)} + \frac{1}{2} \text{ trace } \mathbf{G}^2$$

$$+ (C_0 + E_1^{(n)} + \frac{1}{2} \text{ trace } \mathbf{G})^2, \qquad (179b)$$

and higher moments, though more laboriously determined, are calculated in the same fashion from (172), *viz.*, (179a), (179b). In a similar manner we can determine the *semi-invariants*, or *cumulants*, L_m from the defining relation

$$F_1(i\xi)_x = e^{\log F_1(i\xi)x} = \exp \left\{ \sum_{m=1}^\infty (i\xi)^m L_m/m! \right\}, \qquad (180)$$

as is well-known, $L_1 = \bar{x}$, $L_2 = \overline{x^2} - \bar{x}^2$, etc.

A modification of (162a), (162b) which is useful in our present study of narrow-band models is

$$y = \log \Lambda_n(\mathbf{z}_1, \mathbf{z}_2);$$

$$\log \Lambda_n(\mathbf{z}_1, \mathbf{z}_2) = C_0' + \frac{1}{2}\tilde{\mathbf{z}}_1\mathbf{C}_2'\mathbf{z}_1 + \frac{1}{2}\tilde{\mathbf{z}}_2\mathbf{C}_2'\mathbf{z}_2, \qquad (181)$$

where now $\mathbf{z}_1$ and $\mathbf{z}_2$ are normal random vectors with zero means and *identical* variance matrices $\mathbf{F}$. Furthermore, $\mathbf{z}_1$ and $\mathbf{z}_2$ are statistically independent, so that their joint distribution density is

$$W_{2n}(\mathbf{z}_1, \mathbf{z}_2) = W_n(\mathbf{z}_1)W_n(\mathbf{z}_2)$$

$$= \frac{\exp(-\frac{1}{2}\tilde{\mathbf{z}}_1\mathbf{F}^{-1}\mathbf{z}_1 - \frac{1}{2}\tilde{\mathbf{z}}_2\mathbf{F}^{-1}\mathbf{z}_2)}{(2\pi)^n\{\det \mathbf{F}\}}, \qquad (182)$$

[see (115)]. The characteristic function for y, (181), is easily found by inspection of (168a) or (172), if we observe that now $\bar{\mathbf{z}} = 0$, $\mathbf{C}_1 = 0$, and $\mathbf{z}_1$, $\mathbf{z}_2$ are independent. The result is

$$F_1(i\xi)_y = e^{i\xi C_0'}\{\det(\mathbf{I} - i\xi\mathbf{FC}_2')\}^{-1}$$

$$= e^{i\xi C_0'} \prod_{j=1}^n (1 - i\xi\lambda_j')^{-1}, \qquad (183)$$

where the λ_j' are the n distinct eigenvalues of the matrix $\mathbf{G}' = \mathbf{FC}_2'$, (see Table II). Unlike the more general case (172) considered above, we can here obtain the distribution density of y without recourse to approximations, since the singularities of $F_1(i\xi) y$ occur at $\xi = -i/\lambda_j'$ and are all simple. Contour integration applied to (183) gives directly

$$W_1(y) = \int_{-\infty}^\infty e^{-i\xi(x-C_0')} \prod_{j=1}^n (1 - i\xi\lambda_j')^{-1} \frac{d\xi}{2\pi}$$

$$= \sum_{k=1}^{n+} \frac{e^{-(y-C_0')/\lambda_k'}}{\lambda_k'} \prod_{j=1}^{n+} {}^{(j \neq k)} (1 - \lambda_j'/\lambda_k')^{-1},$$

$$\left. \begin{array}{c} \\ \\ y > C_0' \\ \\ \\ \end{array} \right\} \qquad (184)$$

$$= \sum_{l=1}^{n-} \frac{e^{-(y-C_0')/\lambda'_l}}{\lambda_l'} \prod_{j=1}^{n-} {}^{(j \neq l)} (1 - \lambda_j'/\lambda_l')^{-1},$$

$$y < C_0'$$

where $n^+ + n^- = n$, and n^+ and n^- are respectively the number of positive and negative eigenvalues of $\mathbf{G}' = \mathbf{FC}_2'$. With continuous sampling, $\mathbf{G}'$ is restricted to have at most a finite number of negative eigenvalues, while $n^+ \to \infty$. The relations (183), (184) apply here also under these conditions. In our present work, (cf. Table II), all

genvalues are positive. Relations similar to (183), (184) were obtained originally by Kac and Siegert [5] in their discussion of the problem of distributions after square-law rectification and linear filtering.

Weak-Signal Expansions

In threshold operations we can always apply the trace method and that of steepest descents to obtain asymptotic expressions for the distribution densities $W_1(x)$ and $W_1(y)$ above. It is instructive first, however, to examine the characteristic functions (183), and (168a), (168b), and (172) when $\mathbf{C}_1 = 0 = \bar{z}$. With the aid of the basic identity we have now

$$F_1(i\xi)_x = \frac{e^{i\xi C_0}}{\sqrt{\det (\mathbf{I} - i\xi\mathbf{G})}} = e^{i\xi C_0} \prod_{j=1}^n (1 - i\xi\lambda_j)^{-1/2}$$

$$= \exp\left[i\xi C_0 + \sum_{m=1}^\infty \frac{(i\xi)^m}{2m} \text{trace } \mathbf{G}^m \right]$$

$$(\mathbf{C}_1 = \bar{z} = 0), \qquad (185a)$$

$$F_1(i\xi)_y = \frac{e^{i\xi C_0{}'}}{\det (\mathbf{I} - i\xi\mathbf{G}')} = e^{i\xi C_0{}'} \prod_{j=1}^n (1 - i\xi\lambda_j')^{-1}$$

$$= \exp\left[i\xi C_0' + \sum_{m=1}^\infty \frac{(i\xi)^m}{m} \text{trace } (\mathbf{G}')^m \right]. \qquad (185b)$$

From the defining relation (180) for the semi-invariants we can write at once

$$L_{1(x)} = C_0 + \tfrac{1}{2} \text{trace } \mathbf{G};$$

$$L_{m(x)} = \frac{(m-1)!}{2} \text{trace } \mathbf{G}^m, \qquad (m \geq 2); \qquad (186a)$$

and

$$L_{1(y)} = C_0' + \text{trace } \mathbf{G}';$$

$$L_{m(y)} = (m-1)! \, \text{trace } (\mathbf{G}')^m, \qquad (m \geq 2). \qquad (186b)$$

When the sampling is continuous in $(0, T)$, $\det (\mathbf{I} - i\xi\mathbf{G})$ etc. becomes $\mathfrak{D}_T(-i\xi)$, and trace $\mathbf{G}^m$, trace $(\mathbf{G}')^m$ are replaced by the appropriate iterated kernels $B_m^{(T)}$; see (149a), (149b). Similar remarks apply for the (semi-) infinite observation period $(0, \infty)$.

The threshold distributions of x and y are (asymptotically) normal, with correction terms as indicated below. To obtain this result, let us again retain only terms (ξ, ξ^2) in the exponents of the characteristic function, and develop the rest in a series. Using (180) we may accordingly write in general,

$$F_1(i\xi) = \exp\left\{ \sum_{m=1}^\infty \frac{(i\xi)^m}{m!} L_m \right\}$$

$$= \exp \{i\xi L_1 - \xi^2 L_2/2\} \left\{ 1 + \frac{L_3(i\xi)^3}{3!} \right.$$

$$+ \left[\frac{L_4(i\xi)^4}{4!} + \frac{L_3^2(i\xi)^6}{72} \right]$$

$$\left. - \left[\frac{+L_5(i\xi)^5}{5!} + \frac{L_3 L_4(i\xi)^7}{144} + \frac{L_3^3(i\xi)^9}{6^4} \right] + \cdots \right. \qquad (187)$$

which is an expansion of the Edgeworth type,[41] chosen because it leads to truly asymptotic representations of $W_1(x)$, $W_1(y)$.

From the fact that

$$\int_{-\infty}^\infty (i\xi)^k \exp \{-i\xi A - \xi^2 B/2\} \frac{d\xi}{2\pi}$$

$$= B^{-k/2-1/2} \frac{d^k}{dz^k} \frac{e^{-z^2/2}}{\sqrt{2\pi}}, \qquad z = A/\sqrt{B}, \qquad (188)$$

we obtain from (164)

$$W_1 \simeq \frac{1}{\sqrt{L_2}} \{\phi^{(0)}(z) + C_3\phi^{(3)}(z) + [C_4\phi^{(4)}(z) + C_6\phi^{(6)}(z)]$$

$$+ [C_5\phi^{(5)}(z) + C\phi^{(7)}(z) + C_9\phi^{(9)}(z)] + \cdots , \qquad (189)$$

where now [16]

$$\phi^{(b)}(z) \equiv \frac{d^k}{dz^k} \frac{e^{-z^2/2}}{\sqrt{2\pi}}, \quad \text{and} \quad z = \frac{x - L_1}{L_2^{1/2}} \quad \text{with}$$

$$C_3 = -L_3/3! \, L_2^{3/2}; \qquad C_4 = L_4/4! \, L_2^2;$$

$$C_5 = -L_5/5! \, L_2^{5/2}; \qquad C_6 = L_3^2/72 L_2^3 \qquad (189a)$$

$$C_7 = -L_3 L_4/144 L_2^{7/2}; \qquad C_9 = -L_3^3/6^4 L_2^{9/2},$$

and (186a), (186b) apply respectively here for our particular characteristic functions (185a), (185b). Again, for continuous sampling, the appropriate iterated kernels take the place of trace $\mathbf{G}^m$, etc. Note that in this form, and under these conditions, it is not necessary to calculate the eigenvalues of $\mathbf{G}$, or $\mathbf{G}'$. For $W_1(y)$, of course, exact results can be found, (184), but for $W_1(x)$, (185a), this is not possible, and our only approach is as given above, for threshold performance, and is not valid when the weak signal condition is removed.

Appendix III

Trace Calculations and Iterated Kernels in the Weak-Signal Theory

A. Some General Relations

To obtain the bias and the semi-invariants with the coefficients C_l, cf. (185a)-(189), required in the threshold theory, we must consider the dependence of these quantities on the input signal-to-noise (power) ratio a_0^2 and develop the results in suitable ascending series in this parameter. We find for the situation of discrete sampling (with background noise of finite total intensity) that the matrix $\mathbf{G}$ takes the following forms:

$$(G)_{\text{bias}} = a_0^2 \mathbf{D}; \qquad \mathbf{G}_N = \mathbf{I} - (\mathbf{I} + a_0^2 \mathbf{D})^{-1};$$

$$\mathbf{G}_{S+N} = a_0^2 \mathbf{D}; \qquad \mathbf{D} = \mathbf{k}_S \mathbf{k}_N^{-1}. \qquad (190)$$

The quantity of interest here is trace $\mathbf{G}^m$ $(m \geq 1)$. Let us accordingly consider the case of noise alone first (as

---

[41] Bibliography [7], (17.7).

indicated by the subscript N), expanding the matrix $\mathbf{G}_N^m$. Taking the trace gives

$$\text{trace } \mathbf{G}_N^m = \sum_{l=0}^{m} (-1)_m^l C_l$$

$$\cdot \sum_{q=1}^{\infty} \frac{(-1)^q (q+l-1)!}{q!(l-1)!} a_0^{2q} \text{ trace } \mathbf{D}^q, \quad (191)$$

$$|a_0^{2q} \text{ trace } \mathbf{D}^q| < 1,$$

where $_mC_l$ is the binomial coefficient $m!/(m-l)!l!$. This can be written more compactly as

$$\text{trace } \mathbf{G}_N^m = \sum_{q=1}^{\infty} (-1)^q b_{m,q} a_0^{2q} \text{ trace } \mathbf{D}^q, \quad (192a)$$

with

$$b_{m,q} = \sum_{l=0}^{\infty} {}_mC_l \ _{(q+l-1}C_q) \ (-1)^l$$

$$= (-1)^m (q-1)(q-2) \cdots (q-m+1)/(m-1)!$$

$$= (-q)_m/q(m-1)!. \quad (192b)$$

In terms of the eigenvalues $\lambda_i^{(D)}$ of $\mathbf{D}$, trace $\mathbf{D}^q$ is alternatively $\sum_i^n (\lambda_i^{(D)})^q$; see (148). Similarly, for signal and noise, we have

$$\text{trace } \mathbf{G}_{S+N}^m = a_0^{2m} \text{ trace } \mathbf{D}^m = a_0^{2m} \sum_{j=1}^{n} [\lambda_j^{(D)}]^m. \quad (193)$$

When continuous sampling is employed in the general case ($\mathbf{D} = \mathbf{k}_S \mathbf{k}_N^{-1}$), (192a) is not a convenient form, but for the problem of a white noise background ($\mathbf{D} = \mathbf{k}_S$) it turns out that these various expressions for the trace operations go over into comparatively simple expressions involving the iterated kernels of the signal process. To see this, we recall that the mean intensity of the background noise is now $\lim_{B\to\infty} W_{0N}B = \lim_{n\to\infty} (n/2T) W_{0N}$, where W_{0N} is the spectral density of the interference, so that for this white noise

$$\lim_{n\to\infty} \{a_0^{2m} \text{ trace } \mathbf{D}^m\} = \lim_{n\to\infty} \left\{ \left(\frac{2\psi_S}{W_{0N}}\right)^m \left(\frac{T}{n}\right)^m \text{ trace } k_S^M \right\},$$

$$(\Delta t = T/n),$$

$$= \psi_S^m \lim_{n\to\infty} \left\{ \left(\frac{2}{W_{0N}}\right)^m \Delta t^m \sum_{l_1 \cdots l_m} (k_S)_{l_1 l_2} \cdots (k_S)_{l_m l_1} \right\}$$

$$= 2^m \sigma_0^{2m} \{T^{-m}[B_m^{(T)}]_S\}, \quad (194)$$

with $\sigma_0^2 \equiv \psi_S T/W_{0N}$ an input signal-to-noise (intensity ratio), and $(B_m^{(T)})_S$ the iterated kernel for the signal, e.g.,

$$(B_m^{(T)})_S \quad (195)$$

$$= \int \cdots \int_0^T k_S(t_1, t_2) k_S(t_2, t_3) \cdots k_S(t_m, t_1) \, dt_1 \cdots dt_m.$$

If the signal process is stationary, the symmetrical kernels $k_S(t_i, t_k)$ can be written $k_S(|t_i - t_k|)$; note that $(B_m^{(T)})_S T^{-m}$ is dimensionless.

B. The Bias

With Appendix III-A in mind we can write the bias in a number of alternative ways, of varying utility for computation. Returning for the moment to the general case $\mathbf{D} = \mathbf{k}_S \mathbf{k}_N^{-1}$, we have

$$\Gamma_0 = \log \mu - \tfrac{1}{2} \log \det (\mathbf{I} + \mathbf{G}_{\text{bias}})$$

$$= \log \mu - \tfrac{1}{2} \log \det (\mathbf{I} + a_0^2 \mathbf{D})$$

$$= \log \mu + \log \prod_{j=1}^{n} (1 + a_0^2 \lambda_j^{(D)})^{-1/2}. \quad (196)$$

In terms of the identity (155) this becomes

$$\Gamma_0 = \log \mu + \sum_{m=1}^{\infty} \frac{(-1)^m}{2m} a_0^{2m} \text{ trace } \mathbf{D}^m, \quad (197)$$

and the series is convergent, provided $a_0^2 |\lambda_1^{(D)}| < 1$ where $\lambda_1^{(D)}$ is the largest eigenvalue of $\mathbf{D}$, cf. Appendix II-A.

For continuous sampling in $(0, T)$ we have from (161a) (161b)

$$\Gamma_0 = \log u - \tfrac{1}{2} \log \mathfrak{D}_T(a_0^2)$$

$$= \log \mu + \sum_{m=1}^{\infty} \frac{(-1)^m}{2m} a_0^{2m} [B_m^{(T)}]_D, \quad (198)$$

where $(B_m^{(T)})_D$ is the iterated kernel for $\mathbf{D} = \mathbf{k}_S \mathbf{k}_N^{-1}$, namely

$$[B_m^{(T)}]_D \quad (198a)$$

$$= \int \cdots \int_0^T D(t_1, t_2) D(t_2, t_3) \cdots D(t_m, t_1) \, dt_1 \cdots dt_m.$$

The second relation in (198) is equivalent to the logarithm of the Fredholm determinant (which is absolutely convergent for all $a_0^2 \geq 0$), as long as $a_0^2 |[\lambda_1^{(T)}]_D|^{-1} < 1$, in which $(\lambda_i^{(T)})_D$ is now the largest eigenvalue of (152a) with $G(t, \tau)$ replaced by $D(t, \tau)$. In the case of a white noise background, similar remarks apply to the appropriately modified version of (198), namely

$$\Gamma_0 = \log \mu + \sum_{m=1}^{\infty} 2^{m-1} \frac{(-1)^m}{m} \sigma_0^{2m} (T^{-m}[B_m^{(T)}]_S), \quad (199)$$

(194), (195).

C. Narrow-Band Signals

All of the above carries over directly for the alternative treatment (see Section II) of narrow-band signals. Instead of (190) we have

$$(\mathbf{G}')_{\text{bias}} = a_0^2 \mathbf{D}'; \quad \mathbf{G}_N' = \mathbf{I} - (\mathbf{I} + a_0^2 \mathbf{D}')^{-1};$$

$$\mathbf{G}_{S+N}' = a_0^2 \mathbf{D}'; \quad \mathbf{D}' = \mathbf{k}_S'(\mathbf{k}_N')^{-1}, \quad (200)$$

where $\mathbf{k}_S'$, $\mathbf{k}_N'$ are given in Section VII. The semi-invariants are modified according to (186b) while the bias is now

$$\Gamma_0' = \log \mu - \log \det (\mathbf{I} + a_0^2 \mathbf{D}')$$

$$= \log \mu + \log \prod_{j=1}^{n} \{1 + a_0^2 \lambda_j^{(D')}\}^{-1}, \quad (201)$$

while (198), (199) become alternatively

$$\Gamma_0' = \log \mu + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} a_0^{2m} [B_m^{(T)}]_{D'}, \quad \text{with} \quad (201a)$$

$$\Gamma_0' \big|_{\text{white}} = \log \mu + \sum_{m=1}^{\infty} \frac{\sigma_0^{2m}}{m} (-1)^m \{T^{-m}[B_m^{(T)}]_{S'}\}, \quad (201b)$$

is last for white noise.[42] The eigenvalues and integral
quations for them have the same form as for the general
pproach, but, of course, may be quite different in detail.

. *RC-Noise Signal (White Noise Background and Con-
uous Sampling)*

Here the signal process is obtained by passing originally
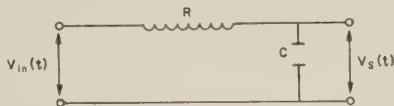nite noise of spectral density W_{0S} through the RC

Fig. 8—An RC-noise signal process $V_s(t)$.

ter of Fig. 8, and taking the output across the condenser
. The spectrum and covariance functions are here

$$
\left.
\begin{array}{ll}
{}_s(f) = \dfrac{4\psi_S \omega_F}{\omega^2 + \omega_F^2} ; & K_s(t) = \psi_S e^{-\omega_F|t|}, \\[2mm]
\psi_S = W_{0S}\omega_F/4; & \omega_F = (RC)^{-1}; \quad \omega = 2\pi f.
\end{array}
\right\} \quad (202)
$$

or the normalized kernel $k_S(t) = e^{-\omega_F|t|}$ it can be
own after considerable, though direct, calculation that
e iterated kernels $(B_M^{(T)})_S$ here become specifically for
e first five orders

$$
{}_{,RC}^{(T)} = T; \qquad B_{2,RC}^{(T)} = T^2\left\{\frac{2\lambda + e^{-2\lambda} - 1}{2\lambda^2}\right\};
$$

$$
{}_{3,RC}^{(T)} = T^3 \cdot \frac{3}{2}\left\{\frac{(\lambda - 1) + (\lambda + 1)e^{-2\lambda}}{\lambda^3}\right\};
$$

$$
{}_{4,RC}^{(T)} = T^4\left\{\frac{2e^{-2}}{\lambda^2} + \frac{10e^{-2\lambda} + 5}{2\lambda^3} + \frac{28e^{-2\lambda} + e^{-4\lambda} - 29}{8\lambda^4}\right\};
$$

$$
{}_{5,RC}^{(T)} = 5T^5\left\{\frac{e^{-2\lambda}}{3\lambda^2} + \frac{3e^{-2\lambda}}{2\lambda^3} + \frac{20e^{-2\lambda} + e^{-4\lambda} + 7}{8\lambda^4}\right.
$$
$$
\left. + \frac{12e^{-2\lambda} + e^{-4\lambda} - 13}{8\lambda^5}\right\}, \qquad (203)
$$

here $\lambda \equiv \omega_F T$. If λ is large (the case of usual interest in
reshold theory), we may approximate the $B_{M,RC}^{(T)}$ above
r all orders, omitting the exponentials and retaining
ly the smallest powers of λ^{-k} in each case. We find then
at $\lambda^{-1+m}T^{-m}B_{m,RC}^{(T)}$ is given approximately as the
efficient of λ^{-m+1}, $m \geq 1$, in the expansion of $(1 - 2/\lambda)^{-1/2}$,

$$
large: \quad T^{-m}B_{m,RC}^{(T)} \cong \frac{2^{m-1}(1/2)_{m-1}}{(m-1)!}\lambda^{1-m} + 0(\lambda^{-m}),
$$
$$
(m \geq 1), \qquad (204)
$$

., while λ is normally $0(5)$ or so, for reasonable approxi-
ations (and is exact for $m = 1$).

On the other hand, when λ is small compared to unity,
rresponding to short observation times, we can derive

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some useful results for $B_{m,RC}^{(T)}$ by expanding the kernels,
i.e., $k_S(t_i - t_k) = 1 - \omega_F|t_i - t_k| + (\omega_F^2/2)(t_i - t_k)^2 + \cdots$.
After some manipulation, we obtain

$$
B_{m,RC}^{(T)} \doteq T^m - \omega_F mT^{m-2}\iint_0^T |t_i - t_{i+1}|\, dt_i\, dt_{i+1}
$$

$$
+ \frac{\omega_F^2}{2}\left\{mT^{m-2}\iint_0^T |t_i - t_{i+1}|^2\, dt_i\, dt_{i+1}\right.
$$

$$
+ 2mT^{m-3}\int\cdots\int_0^T |t_i - t_{i+1}|
$$
$$
\cdot |t_{i+1} - t_{i+2}|\, dt_i\, dt_{i+1}\, dt_{i+2}
$$

$$
+ m(m - 3)T^{m-4}\int\cdots\int_0^T |t_i - t_{i+1}|
$$
$$
\left.\cdot|t_k - t_{k+1}|\, dt_i\, dt_{i+1}\, dt_k\, dt_{k+1}\right\} + \cdots, \qquad (204)
$$

and carrying out the integration gives finally

$$
T^{-m}B_{m,RC}^{(T)} \doteq 1 - \frac{m\lambda}{3}
$$
$$
+ \frac{\lambda^2}{2!}\left[\frac{17m}{60} + \frac{m(m - 3)}{9}\right] + 0(\lambda^3), \quad (m \geq 2). \qquad (205)
$$

Inserting (204) and (205) into the expression (199) for
the bias, we may sum the series to obtain

$$
\Gamma_{0,RC}| \simeq \log \mu - \frac{\lambda}{2}\left\{\sqrt{1 + \frac{4\sigma_0^2}{\lambda}} - 1\right\} + 0(\lambda^{-2}, \sigma_0^0) \quad (206a)
$$

$$
\Gamma_{0,RC}| \doteq \log \mu - \sigma_0^2\left\{\frac{\log(1 + 2\sigma_0^2)}{2\sigma_0^2} + \frac{\lambda}{3}\left(\frac{2\sigma_0^2}{1 + 2\sigma_0^2}\right)\right.
$$
$$
\left. - \left(\frac{\lambda^2}{360}\right)\left(\frac{2\sigma_0^2}{1 + 2\sigma_0^2}\right)\left(\frac{31 + 22\sigma_0^2}{1 + 2\sigma_0^2}\right)\right\} + 0(\lambda^3, \sigma_0^2), \qquad (206b)
$$

respectively. In a similar way we can use these approxi-
mations to determine the semi-invariants in the case of
white noise, with the aid of (194) applied to (192a),
(193). Using (204), and summing the various series in m,
we get

$$
\left.
\begin{array}{l}
L_{m,RC}^{(N)} \cong (1/2)_{m-1}\sigma_0^2(4\sigma_0^2/\lambda)^{m-1}/(1 + 4\sigma_0^2/\lambda)^{m-1/2}, \\[2mm]
\hspace{5cm} (m \geq 2) \\[2mm]
L_{m,RC}^{(S+N)} \cong (1/2)_{m-1}\sigma_0^{2m}2^{2m-2}/\lambda^{m-1},
\end{array}
\right\} \quad \begin{array}{c}(207a)\\[4mm](207b)\end{array}
$$

for noise alone and for signal and noise, where λ is large,
i.e., the observation period is comparatively long. For
$m = 1$, the above still applies if we add C_0; see (186a).

*E. Narrow-Band, LRC-Noise Signal (White Noise Back-
ground and Continuous Sampling)*

Again, white noise of spectral intensity W_{0S} is passed
through a linear filter, to generate the signal process. The
filter in question is now of the LRC type, shown in Fig.
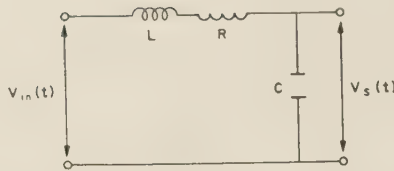
Fig. 9—An LRC-noise signal process $V_s(t)$.

9, and the associated spectrum and covariance function of the signal ensemble are respectively

$$\mathcal{W}_S(f) = \frac{W_{0S}\omega_0^4}{4\omega_F^2\omega^2[1 + (\omega^2 - \omega_0^2)^2/4\omega^2\omega_F^2]} \qquad (208a)$$

$$K_S(t) = \psi_S e^{-\omega_F|t|}\left(\cos\omega_1 t + \frac{\omega_F}{\omega_1}\sin\omega_1|t|\right), \qquad (208b)$$

where

$$\omega_1 = \sqrt{\omega_0^2 - \omega_F^2}; \qquad \omega_0^2 = 1/LC; \qquad \omega_F = R/2L;$$

$$(RC = 2\omega_F/\omega_0^2); \qquad \psi_S = W_{0S}\omega_0^2/8\omega_F. \qquad (208c)$$

[Note that $\omega_F|_{RC}$, (202), and $\omega_F|_{LRC}$, (208c), are different quantities]. In the high-Q cases we are interested in here, (208) becomes

$$\mathcal{W}_S(f) = 2\psi_S\omega_F^{-1}\left[1 + \left(\frac{\omega - \omega_0}{\omega_F}\right)^2\right]^{-1};$$

$$K_S(t) \doteq \psi_S e^{-\omega_F|t|}\cos\omega_0 t, \qquad (209)$$

since $Q(\equiv \omega_0/2\omega_F)$ is now taken to be very large. If we insert this expression for $k_S = K_S\psi_S^{-1}$ into the various iterated kernels (195), etc., we find that only terms in the integrand that do not contain $\cos\omega_0 t$ make a significant contribution. In fact, we easily establish that, *in form*,

$$B_{m,LRC}^{(T)} \doteq 2^{1-m}B_{m,RC}^{(T)}, \qquad (210)$$

where we observe, of course, that $\lambda_{RC} = (\omega_F)_{RC}T$ and $\lambda = (\omega_F)_{LRC}T$ are usually different magnitudes, represented in both instances by λ, while it is normally clear from the context which is meant. Note here that $k_S'(|t|)_0 = e^{-\omega_F|t|}$, is formally identical with $k_S(t)$ of the RC-noise signal discussed in Appendix III-D.

Modifying (204) accordingly, and repeating the steps outlined above for the RC-noise signal, we get in straightforward fashion

$$\Gamma_{0,LRC} \simeq \log\mu - \lambda(\sqrt{1 + 2\sigma_0^2/\lambda} - 1) + 0(\lambda^{-2}, \sigma_0^0) \qquad (211a)$$

$$\Gamma_{0,LRC} = \log\mu - \sigma_0^2\left\{\frac{\log(1 + \sigma_0^2)}{\sigma_0^2} + \frac{\lambda}{6}\left(\frac{\sigma_0^2}{1 + \sigma_0^2}\right)\right.$$

$$\left. - \frac{\lambda^2}{720}\left(\frac{\sigma_0^2}{1 + \sigma_0^2}\right)\left(\frac{31 + 11\sigma_0^2}{1 + \sigma_0^2}\right) + 0(\lambda^3, \sigma_0^0)\right\}, \qquad (211b)$$

respectively for large and small $\lambda(= \lambda_{LRC}$ here). Similarly, from (186a), (186b) the semi-invariants are simply,

$$L_{m,LRC}^{(N)} = 2L_{m,RC}^{(N)}; \quad L_{m,LRC}^{(S+N)} = 2L_{m,RC}^{(S+N)} \quad (m \geq 2), \qquad (212)$$

while for $m = 1$ we have, instead, $L_{1,LRC}^{(N)} = C_0' + 2 L_{1,RC}^{(N)}$; $L_{1,LRC}^{(S+N)} = C_0' + 2 L_{1,RC}^{(S+N)}$. Again, provided the signal is narrow-band, (207) can be used in (212) to give

the desired semi-invariants directly for the LRC case.

Since σ_0^2 is proportional to T, it is often convenient to use another signal-to-noise ratio, called the *effective* signal-to-noise (intensity) ratio, defined by

$$\sigma_0^2 \equiv \lambda\sigma_e^2; \qquad \sigma_e^2 \equiv \psi_S/W_{0N}\omega_F;$$

$$[\omega_F = (\omega_F)_{RC}, \quad \text{or} \quad (\omega_F)_{LRC}]. \qquad (213)$$

Then, for the specific signals of Appendix III-D and III-E we can compute the semi-invariants and related C's of the Edgeworth expansions (189), in the cases of long integration times ($\lambda > 5$).

APPENDIX IV
SOLUTIONS OF THE INTEGRAL EQUATION FOR CONTINUOUS SAMPLING, WITH RATIONAL SPECTRA, IN A WHITE NOISE BACKGROUND

A. A General Solution of the Integral Equation

The integral equation we have to consider here takes the general form (for stationary processes)

$$\int_0^T K_S(t - u)z_T(u)\,du + Az_T(t)$$

$$= B\int_0^T K_S(t - u)V(u)\,du, \qquad 0 < t < T), \qquad (214)$$

where A and B are real, $A \geq 0$, $|B| > 0$. {For our particular problem $A = W_{0N}/2$, $B = (W_{0N}/2)^{-1}$, so that A has the dimensions of [amplitude^2/frequency], and since K_S is [amp^2], V is [amp], we see that z_T has the dimensions [amp^{-1} freq]. In what follows, however, we shall let A and B be constants, not necessarily related by $B = A^{-1}$, but with the appropriate dimensions, as determined by (214).} It is now postulated that:

1) $z_T(t)$, $V(t) = 0$ outside $(0-, T+)$;

2) $K_S(t - u) = K_S(|t - u|)$

$$= \sum_{n=1}^N a_n e^{-b_n|t-u|}, \qquad \text{Re}(b_n) > 0.$$

This is a form of "lumped-constant" noise, yielding a rational spectrum; the covariance function K_S is real, so that the a_n's, b_n's occur in complex conjugate pairs ($N \geq 2$).

Thus, we may write (214), using the continuity property of K_S, as

$$\int_0^T K_S(t - u)[z_T(u) - BV(u)]\,du + Az(t)$$

$$= \begin{cases} \sum_1^N A_n^{(+)}e^{-b_n(t-T)}, & t > T \\ 0, & 0 < t < T \\ \sum_1^N A_n^{(-)}e^{b_n t}, & t < 0 \end{cases} \qquad (215)$$

where the $A_n^{(\pm)}$ (with the dimensions [ampl^2]) are to be found from the requirement that $z(t) = 0$ outside $(0, T)$.

Let us now write

$$_T(p)_z = \int_0^T e^{-pt} z_T(t)\, dt;$$

$$S_T(p)_V = \int_0^T e^{-pt} V(t)\, dt, \qquad (216)$$

d

$$_S(f) = \frac{W_{0S}}{2} Y(i\omega)^2, \qquad \omega = 2\pi f;$$

$$_S(p/2\pi i) = \frac{W_{0S}}{2} Y(p) Y(-p)$$

$$= 2 \int_{-\infty}^{\infty} e^{-pt} K_S(t)\, dt, \qquad (217a)$$

is last by the Wiener-Khintchine theorem, where also
e inverse relation is

$$K_S(t) = \frac{1}{2} \int_{-\infty i}^{\infty i} e^{pt} \mathcal{W}_S(p/2\pi i) \frac{dp}{2\pi i}. \qquad (217b)$$

arrying out the indicated operations for the specific
ernel $K_S(t)$, above (215), gives

$$\mathcal{W}_S(p/2\pi i) = 4 \sum_{n=1}^{N} \frac{a_n b_n}{b_n^2 - p^2}. \qquad (218)$$

king the Fourier transform of both sides of (215), and
serving that (for $p = 2\pi i f$)

$$_{\infty} K_S(t - u) e^{-pt}\, dt = e^{-pu} \int_{-\infty}^{\infty} e^{-px} K_S(x)\, dx$$

$$= \frac{e^{-pu}}{2} \mathcal{W}_S(p/2\pi i) \qquad (219)$$

e obtain in a straightforward way the transformed
rsion of (215), namely,

$$_(p)_z = [2A + \mathcal{W}_S(p/2\pi i)]^{-1} \Big\{ 2 \sum_{n=1}^{N} \left(\frac{A_n^+ e^{-pT}}{b_n + p} + \frac{A_n^{(-)}}{b_n - p} \right)$$

$$+ BS_T(p)_V \mathcal{W}_S(p/2\pi i) \Big\}. \qquad (220)$$

We now observe that there are $2N$ roots of $2A + _S(p_{\pm k}) = 0$, $k = 1, 2, \cdots, N$, so that writing

$$H(p) \equiv [2A + \mathcal{W}_S(p/2\pi i)]^{-1} \qquad (221)$$

e have

$$_{\infty i}^{\infty i} e^{pt} \mathcal{W}_S(p/2\pi i) H(p) \frac{dp}{2\pi i}$$

$$\equiv \sum_{n=1}^{N} h_n \exp\{-p_n^{|t|}\} = h(|t|). \qquad (222)$$

ere the p_n, $[n = 1, \cdots N]$, are the n roots of $H(p)^{-1}$ with
sitive real parts. That h is a function of $|t|$ follows from
e fact that the real parts of the poles of $\mathcal{W}_S$, and $H(p)$,
e symmetrically located on either side of the imaginary
is. Further, we have assumed here that none of the

roots of $H(p)^{-1}$ is multiple, in accordance with 2) above.
In fact, we may write

$$\mathcal{W}_S(p/2\pi i) \equiv \mathfrak{X}_N(p)/\Phi_N(p) \qquad (223a)$$

$$\therefore H(p) = \Phi_N(p)/\{2A\Phi_N(p) + \mathfrak{X}_N(p)\} \equiv \Phi_N(p)/\psi_N(p);$$

$$\Psi_N(p) \equiv 2A\Phi_N(p) + \mathfrak{X}_N(p) = \Psi_N(-p). \qquad (223b)$$

In terms of (218) we have specifically

$$\mathcal{W}_S(p/2\pi i) \qquad (224a)$$

$$= 4 \sum_{n=1}^{N} a_n b_n \prod_{j=1}^{N}{}^{(j \neq n)} (b_i^2 - p^2) \Big/ \prod_{n=1}^{N} (b_n - p)(b_n + p),$$

so that

$$\mathfrak{X}_N(p) = 4 \sum_{n=1}^{N} a_n b_n \prod_{j=1}^{N'} (b_i^2 - p^2);$$

$$\Phi_N(p) = \prod_{n=1}^{N} (b_n^2 - p^2). \qquad (224b)$$

Writing c_n for p_n, (222) *et seq.*, let us define (with attention
to dimensionality)

$$\Psi_N(p) \equiv A \prod_{n=1}^{N} (c_n^2 - p^2) = 2A \prod_{n=1}^{N} (b_n^2 - p^2)$$

$$+ 4 \sum_{n=1}^{N} a_n b_n \prod_{j=1}^{N'} (b_i^2 - p^2), \qquad (225)$$

with $c_n \neq b_n$ and $\mathrm{Re}(c_n) > 0$. From (221), (222), (223a),
(223b), we get, with straightforward integration

$$\int_{-\infty i}^{\infty i} e^{pt} \mathfrak{X}_N(p) \Psi_N(p)^{-1} \frac{dp}{2\pi i}$$

$$= 2A^{-1} \sum_{n=1}^{N} \frac{a_n b_n}{c_n} \prod_{j=1}^{N'} \left(\frac{b_i^2 - c_n^2}{c_j^2 - c_n^2} \right) e^{-c_n|t|}, \qquad (226)$$

from which it follows that

$$h_n = \frac{2}{A} \cdot \frac{a_n b_n}{c_n} \prod_{j=1}^{N'} \left(\frac{b_i^2 - c_n^2}{c_j^2 - c_n^2} \right). \qquad (227)$$

At this point we now define

$$G(p) \equiv \mathcal{W}_S(p/2\pi i) H(p)$$

$$= \int_{-\infty}^{\infty} h(|t|) e^{-pt}\, dt, \qquad \mathrm{Re}(p) = 0, \qquad (228)$$

[and subsequently continue $G(p)$ analytically]. Taking
the transform of the last member of (220), we obtain

$$\int_{-\infty i}^{\infty i} G(p) S_T(p)_V e^{pt} \frac{dp}{2\pi i}$$

$$= \int_{-\infty}^{\infty} V_T(t + t') h(|t'|)\, dt'$$

$$= \int_0^T h(|t - t'|) V(t')\, dt'. \qquad (229)$$

Now, since the zeros of $H(p)^{-1}$ are the same as the zeros
of $\Psi_N(p)$, (223a), (223b), and since the poles of the ex-
pressions in the parentheses in (220) are the same as the
corresponding zeros of $\Phi_N(p) = \prod_{n=1}^{N} (b_n^2 - p^2)$, when
we take the Fourier transform of both sides of (220), we

observe that these latter cancel one another, and the only poles of the integrand occur at the zeros of $\Psi_N(p)$, (223b). Thus, with the help of (229), using (222), (223), (224b), (225) we get

$$z_T(t) = 2A^{-1} \int_{-\infty i}^{\infty i} \frac{dp}{2\pi i} e^{pt} \prod_{n=1}^{N} (b_n^2 - p^2)(c_n^2 - p^2)^{-1} \quad (230)$$

$$\cdot \left\{ \sum_{n=1}^{N} \left(\frac{A_n^{(+)} e^{-pT}}{b_n + p} + \frac{A_n^{(-)}}{b_n - p} \right) \right\}$$

$$+ B \sum_{n=1}^{N} h_n \int_0^T V(t') e^{-c_n |t - t'|} \, dt',$$

where for noise shaped by physically realizable networks—here linear, lumped-constant filters—we require that $\mathrm{Re}(b_n)$, $\mathrm{Re}(c_n) > 0$. Notationally, we have also $z_T = V_T = 0$, $(t > T, t < 0)$, while $z_T(t) = z(t)$; $V_T(t) = V(t)$, $(0 < t < T)$.

We are now in a position to evaluate (230). Observing for $t > T$ that the poles of the integrand are simple and occur at $p = -c_n$, and that for $0 < t < T$, (all $t < T$), the poles are at $p = c_n$, while for $t < 0$, they also occur at $p = c_n$, we find that (230) becomes explicitly

$\underline{(t > T)}$: $0 = B \sum_{n=1}^{N} (h_n \int_0^T V(x) e c_n^x \, dx) e^{-c_n t}$

$$+ 2A^{-1} \sum_{n=1}^{N} (A_n^{(+)} d_n^{(N)} e c_n^T) e^{-c_n T}$$

$$+ 2A^{-1} \sum_{n=1}^{N} A_n^{(-)} e_n^{(N)} e^{-c_n T} \quad (231a)$$

$\underline{(0 < t < T)}$: $z(t) = B \sum_{n=1}^{N} h_n \int_0^T V(x) e^{-c_n |t - x|} \, dx$

$$+ 2A^{-1} \sum_{n=1}^{N} (A_n^{(+)} e_n^{(N)} e^{-c_n t}$$

$$+ 2A^{-1} \sum_{n=1}^{N} A_n^{(-)} e_n^{(N)} e^{-c_n t} \quad (231b)$$

$\underline{(t < 0)}$: $0 = B \sum_{n-1}^{N} h_n e^{c_n t} \int_0^T V(x) e^{-c_n x} \, dx$

$$+ 2A^{-1} \sum_{n-1}^{N} (A_n^{(+)} e_n^{(N)} e^{-c_n t}$$

$$+ 2A^{-1} \sum_{n=1}^{N} A_n^{(-)} d_n^{(N)} e^{c_n t} \quad (231c)$$

where $d_n^{(N)}$ and $e_n^{(N)}$ are given by

$$d_n^{(N)} = \frac{\prod_{j=1}^{N} (b_j + c_j) \prod_{j=1}^{N}{}^{(j \neq n)} (b_j - c_j)}{\prod_{j=1}^{N} (c_j + c_n) \prod_{j=1}^{N'} (c_j - c_n)} \quad (232a)$$

$$e_n^{(N)} = \frac{\prod_{j=1}^{N'} (b_j + c_j) \prod_{j=1}^{N} (b_j - c_j)}{\prod_{j=1}^{N} (c_j + c_n) \prod_{j=1}^{N'} (c_j - c_n)}. \quad (232b)$$

Now since (231a), (231c) are each identities, the coefficients of $e^{-c_n t}$, $e^{c_n t}$ must vanish for each n $(= 1, \cdots, N)$, giving us $2N$ equations from which to determine the $2N$ unknown constants $A_n^{(\pm)}$, $(n = 1, \cdots, N)$. Letting

$$\begin{cases} \hat{a}_n \equiv Bh_n \int_0^T V(x) e^{c_n x} \, dx; \quad \hat{a}'_n \equiv Bh_n \int_0^T V(x) e^{-c_n x} \, dx \\ \hat{b}_n \equiv 2d_n^{(N)} e^{c_n T} A^{-1}; \quad\quad\quad \hat{b}'_n \equiv 2e_n^{(N)} e^{-c_n T} A^{-1} \\ \hat{c}_n \equiv 2e_n^{(N)} A^{-1}; \quad\quad\quad\quad\quad \hat{c}'_n \equiv 2d_n^{(N)} A^{-1}, \end{cases} \quad (233)$$

with $X_n \equiv A_n^{(+)}$, $Y_n \equiv A_n^{(-)}$, we can write for each n, from (231a), (231c) the set of equations

$$\left. \begin{array}{c} \hat{a}_n + \hat{b}_n X_n + \hat{c}_n Y_n = 0 \\ \hat{a}'_n + \hat{b}'_n X_n + \hat{c}'_n Y_n = 0 \end{array} \right\} \quad (n = 1, \cdots, N) \quad (234)$$

for which the solutions are

$$X_n \equiv A_n^{(+)} = \frac{\hat{a}_n \hat{c}'_n - \hat{a}'_n \hat{c}_n}{\Delta_n};$$

$$Y_n \equiv A_n^{(-)} = \frac{\hat{a}'_n \hat{b}_n - \hat{a}_n \hat{b}'_n}{\Delta_n}, \quad (n = 1, \cdots, N) \quad (235)$$

provided $\Delta_n \equiv \hat{b}'_n \hat{c}_n - \hat{b}_n \hat{c}'_n \neq 0$, where specifically

$$\Delta_n = 4A^{-2} \left\{ \prod_{j=1}^{N'} \frac{(b_j^2 - c_j^2)^2}{(c_j - c_n)^2} \prod_{j=1}^{N} (c_j + c_n)^{-2} \right\}$$

$$\cdot \{(b_n - c_n)^2 e^{-c_n T} - (b_n + c_n)^2 e^{c_n T}\}, \quad (236)$$

with $b_j \neq c_j$, b_j, c_j distinct, all n. From (232a), (232b), (233), (235), and (236) we get directly for the $2N$ constants $A_n^{(+)}$, $A_n^{(-)}$:

$$A_n^{(+)} = F_n \cdot \left[\frac{(b_n + c_n) \int_0^T V(x) e^{c_n x} \, dx - (b_n - c_n) \int_0^T V(x) e^{-c_n x} \, dx}{(b_n - c_n)^2 e^{-c_n T} - (b_n + c_n)^2 e^{c_n T}} \right] \quad (237)$$

$$A_n^{(-)} = F_n \cdot \left[\frac{e^{c_n T} (b_n + c_n) \int_0^T V(x) e^{-c_n x} \, dx - e^{-c_n T} (b_n - c_n) \int_0^T V(x) e^{c_n x} \, dx}{(b_n - c_n)^2 e^{-c_n T} - (b_n + c_n)^2 e^{c_n T}} \right] \quad (238)$$

where

$$F_n = ABc_n h_n \prod_{j=1}'\left(\frac{c_j^2 - c_n^2}{b_j^2 - c_j^2} \right) = 2Ba_n b_n \prod_{j=1}^{N'} \left(\frac{b_j^2 - c_n^2}{b_j^2 - c_j^2} \right). \quad (239)$$

This, in conjunction with (231b), completes the general solution of (214) and (215), subject to the restriction that $K_S(t)$ possesses a rational spectrum, i.e., K_S is the covariance function of "lumped-constant" noise. When the spectral representation (217a) has multiple poles, an exactly analogous procedure may be employed.

the results may be obtained from (237)-(239) by a suitable passage to the limit, *e.g.*, for double poles:

$$\lim_{b_2 \to b_1} \left\{ \frac{e^{-b_1|t|} - e^{-b_2|t|}}{(b_1 - b_2)} \right\}.$$

Special Forms of the Solution

Some special cases of (214) are readily obtained when is set equal to zero, or when (semi-) infinite observation periods ($T \to \infty$) are allowed. When $A = 0$, we rewrite $_N(p)$ as $B^{-1} \prod_{n=1}^{N} (c_n^2 - p^2)$, (225), where we are now concerned with the roots, c_n, of $\sum_{n=1}^{N} a_n b_n \prod_{i=1}^{N'} (b_i^2 - p^2) = 0$. These then lead us to the corresponding version of 230), from which $z(t)$ is determined as above. A variant of the case ($A = 0$) is

$$\int_{0-}^{T+} K_S(t - u) z(u) \, du = g(t), \quad (0 - < t < T+) \quad (240)$$

here now $g(t)$ is some given function of t in $(0-, T+)$, differentiable to a suitable order. Solutions of (240) in the case of the rational spectra assumed here are available elsewhere [17], [18].

In the limiting situation of semi-infinite observation periods, we find from (237), (238) that

$$\lim_{\to \infty} A_n^{(+)} = F_n \cdot (b_n + c_n)^{-1}$$

$$\cdot \lim_{T \to \infty} \left\{ e^{-c_n T} \int_0^T V(x) e^{c_n x} \, dx \right\} < \infty, \quad (241)$$

here $V(t)$ is assumed to be suitably bounded, for all ≥ 0. We have also

$$\lim_{\to \infty} A_n^{(-)} = F_n \cdot (b_n + c_n)^{-1}$$

$$\cdot \int_0^\infty V(x) e^{-c_n x} \, dx, \quad (< \infty), \quad (242)$$

o that our solution (231b) becomes the Wiener-Hopf elation

$$(t) = B \sum_{n=1}^{N} h_n \left\{ \int_0^\infty V(x) e^{-c_n|t-x|} \, dx \left(\frac{b_n - c_n}{b_n + c_n} \right) \right.$$

$$\left. \cdot e^{-c_n t} \int_0^\infty V(x) e^{-c_n x} \, dx \right\}, \quad t \geq 0 \quad (243)$$

$$= 0, \quad t < 0.$$

RC-Noise Signal

For an RC-noise signal [Fig. 8 and (202)] we find that

$$= 1; \quad a_1 = \psi_S; \quad b_1 = \omega_F; \quad F_1 = c_1 h_1 = 4\psi_S \omega_F / W_{0N};$$

$$h_1 = 2a_1 b_1 / A c_1 = 4\psi_S \omega_F / W_{0N} c_1, \quad (244)$$

where the roots of $2A + \mathcal{W}_S(p/2\pi i)$, (221), are found from

$$2A + 2\psi_S \left(\frac{1}{\omega_F - p} + \frac{1}{\omega_F + p} \right) = 0;$$

$$\therefore c_1 = \omega_F \sqrt{1 + \gamma_0^2}, \quad (\text{Re } c_1 > 0);$$

$$\gamma_0^2 \equiv 4\psi_S / \omega_F W_{0N} = W_{0S} / W_{0N};$$

$$\therefore h_1 = \frac{4\psi_S}{W_{0N}} \sqrt{1 + \gamma_0^2}$$

$$= \left(\frac{W_{0S}}{W_{0N}} \right) \omega_F \sqrt{1 + \gamma_0^2} = \gamma_0^2 \omega_F \sqrt{1 + \gamma_0^2}$$

$$\quad (245a)$$

From this, $F_1 = \omega_F^2 \gamma_0^2$, and

$$d_1^{(1)} = \frac{b_1 + c_1}{2c_1} = \frac{1 + \sqrt{1 + \gamma_0^2}}{2\sqrt{1 + \gamma_0^2}};$$

$$e_1^{(1)} = \frac{b_1 - c_1}{2c_1} = \frac{1 - \sqrt{1 + \gamma_0^2}}{2\sqrt{1 + \gamma_0^2}}. \quad (245b)$$

Returning to (231b), remembering that $B = 2/W_{0N}$, we have here finally for the desired solution

$$z_T(t) = \frac{2\gamma_0^2 \omega_F \sqrt{1 + \gamma_0^2}}{W_{0N}} \int_0^T V(x) e^{-c_1|t-x|} \, dx$$

$$+ \frac{2}{W_{0N}} \left(\frac{1 - \sqrt{1 + \gamma_0^2}}{1 + \gamma_0^2} \right) \quad (246)$$

$$\cdot \{ A_1^{(+)} e^{c_1(t-T)} + A_1^{(-)} e^{-c_1 t} \}, \quad (0 \leq t \leq T),$$

where now specifically

$$A_1^{(+)} = \omega_F \gamma_0^2 \left[\frac{(1 + \sqrt{1 + \gamma_0^2}) \int_0^T V(x) e^{c_1 x} \, dx - (1 - \sqrt{1 + \gamma_0^2}) \int_0^T V(x) e^{-c_1 x} \, dx}{(1 - \sqrt{1 + \gamma_0^2})^2 e^{-c_1 T} - (1 + \sqrt{1 + \gamma_0^2})^2 e^{c_1 T}} \right] \quad (247a)$$

$$A_1^{(-)} = \omega_F \gamma_0^2 \left[\frac{(1 + \sqrt{1 + \gamma_0^2}) e^{c_1 T} \int_0^T V(x) e^{-c_1 x} \, dx - (1 - \sqrt{1 + \gamma_0^2}) \int_0^T V(x) e^{c_1(x-T)} \, dx}{(1 - \sqrt{1 + \gamma_0^2})^2 e^{-c_1 T} - (1 + \sqrt{1 + \gamma_0^2})^2 e^{c_1 T}} \right]. \quad (247b)$$

For semi-infinite observation periods, (246) reduces to the Wiener-Hopf result

$$z_\infty(t) = \frac{2\omega_F \gamma_0^2}{W_{0N}} \left\{ \sqrt{1 + \gamma_0^2} \int_0^\infty V(x) e^{-c_1|t-x|} \, dx \right.$$

$$\left. + \left(\frac{1 - \sqrt{1 + \gamma_0^2}}{1 + \sqrt{1 + \gamma_0^2}} \right) e^{-c_1 t} \int_0^\infty V(x) e^{-c_1 x} \, dx \right\}, \quad t \geq 0. \quad (248)$$

D. LRC-Noise Signal

Here the relations of (208) apply, so that we can write

$$\mathcal{W}_S(f) = \frac{W_{0S} \omega_0^4}{(b_1^2 + \omega^2)(b_2^2 + \omega^2)};$$

$$b_1 = \omega_F - i\omega_1; \quad b_2 = b_1^*; \quad \omega_1^2 = \omega_0^2 - \omega_F^2;$$

$$b_1 + b_2 = 2\omega_F; \quad b_1 b_2 = \omega_0^2;$$

$$b_1^2 + b_2^2 = 2\omega_F^2 - \omega_1^2$$

$$b_1^2 - b_2^2 = -4i\omega_1 \omega_F$$

$$\quad (249)$$

and

$$K_S(t) = a_1 e^{-b_1 |t|} + a_2 e^{-b_2 t};$$

$$a_1 = \frac{W_0^2 \psi_S}{2i\omega_1 b_1}; \qquad a_2 = \frac{W_0^2 \psi_S}{2i\omega_1 b_2}. \qquad (250)$$

The roots of $H(p)^{-1}$, (221), are found from (225) with the aid of the above as solution of

$$p^4 - 2(\omega_F^2 - \omega_1^2)p^2 + \omega_0^4(1 + \gamma_0^2) = 0. \qquad (251)$$

We get finally

$$p_{1,2} = c_{1,2} = (a^2 + b^2)^{1/4} \exp\left\{\pm\frac{i}{2}\left[\tan^{-1}\left(\frac{b}{a}\right) - \frac{\pi}{2}\right]\right\},$$

$$\text{Re } (c_1, c_2 > 0), \qquad (252)$$

where $a \equiv \omega_1^2 - \omega_F^2 \ (> 0 \text{ here})$;

$$b \equiv \omega_0^2 \sqrt{1 + \gamma_0^2 - \left(1 - \frac{1}{2Q^2}\right)^2}, \qquad Q \equiv \omega_0/2\omega_F,$$

with $(a^2 + b^2)^{1/4} = \omega_0(1 + \gamma_0^2)^{1/4}$.

For the high-Q cases, *i.e.*, narrow-band signals, we have

$$c_1 c_2 = W_0(1 + \gamma_0^2)^{1/4}$$

$$\cdot \exp\left\{\pm\frac{i}{2}\left(\tan^{-1}\frac{\sqrt{1 + \gamma_0^2 - (1 - 1/2Q^2)^2}}{(1 - 1/2Q^2)} - \frac{\pi}{2}\right)\right\},$$

$$\text{(exact)} \qquad (253a)$$

$$\doteq \omega_0(1 + \gamma_0^2)^{1/4} \exp\left\{\pm\frac{i}{2}\left(\tan^{-1}\sqrt{1 + \gamma_0^2} - \frac{\pi}{2}\right)\right\}. \quad (253b)$$

Applying the above to (227), (239), we get

$$F_2 = F_1^*; \qquad F_1 = \frac{2\omega_0^2 \psi_S}{iW_{0N}\omega_1}\left(\frac{b_2^2 - c_1^2}{b_2^2 - c_2^2}\right);$$

$$h_1 h_2^* = \frac{2\omega_0^2 \psi_S}{iW_{0N}\omega_1 c_1}\left(\frac{b_2^2 - c_1^2}{c_2^2 - c_1^2}\right), \qquad (254)$$

and since $c_2^* = c_1$, we get finally

$$z_T(t) = \text{Re}\left\{\left(\frac{b_2^2 - c_1^2}{c_2^2 - c_1^2}\right)\frac{8\omega_0^2 \psi_S}{W_{0N}^2\omega_1 c_1 i}\left[\int_0^T V(x)e^{-c_1|t-x|}\,dx\right.\right.$$

$$\left.\left. + (b_1 - c_1)\{(\ )_{n=1}^{(+)} e^{-c_1(T-t)} + (\ )_{n=1}^{(-)} e^{-c_1 t}\}\right]\right\}, \qquad (255)$$

where $(\ )_{n=1}^{(+)} \cdot (\ )_{n=1}^{(-)}$ are respectively the quantities that are coefficients of F_1 in (237), (238), when $n = 1$. For the semi-infinite observation periods we get the Wiener-Hopf solution $(t \geq 0)$

$$z_\infty(t) = \text{Re}\left\{\left(\frac{b_2^2 - c_1^2}{c_2^2 - c_1^2}\right)\frac{8\omega_0^2 \psi_S}{W_{0N}^2\omega_1 c_1 i}\left[\int_0^\infty V(x)e^{-c_1|t-x|}\,dx\right.\right.$$

$$\left.\left. + \left(\frac{b_1 - c_1}{b_1 + c_1}\right)e^{c_1 t}\int_0^\infty V(x)e^{c_1 x}\,dx\right]\right\}. \qquad (256)$$

E. A Related Integral Equation

When the background noise is white, we can obtain an alternative representation of the detector structure which does not involve the received data $V(t)$ explicitly in the resulting integral equation. In matrix form the structure

term is (Section I-D) where for the moment we regar ψ_N as finite (except when passing to the limit $n \to \infty$ o $B \to \infty$)

$$\Phi_n \equiv \psi_N^{-1}\tilde{\mathbf{V}}\mathbf{G}\psi_N\mathbf{V}, \quad \text{with} \quad \mathbf{C} = \mathbf{\Lambda}_N^{-1} - (\mathbf{\Lambda}_N + \mathbf{K}_S)^{-1};$$

$$\mathbf{\Lambda}_N = \left[\frac{nW_{0N}}{2T}\delta_{jk}\right] = [\psi_N\delta_{jk}]. \qquad (257$$

If we write $\mathbf{Z} \equiv \psi_N\mathbf{C}\mathbf{V}$, then $\Phi_n = \psi_N^{-1}\tilde{\mathbf{V}}\mathbf{Z}$ (48) and passag to the limit $(n \to \infty)$ for the continuous cases now gives

$$\lim_{n\to\infty} \Phi_n = \Phi_T = \frac{2}{W_0}\lim_{n\to\infty}\sum_{k=1}^n \frac{T}{n}V(t_k)Z(t_k)$$

$$= \frac{2}{W_{0N}}\int_0^T V(t)Z(t)\,dt. \qquad (258$$

At this point let us write

$$\psi_N\mathbf{C} = \mathbf{I} - (\mathbf{I} + a_0^2\mathbf{k}_S)^{-1} \equiv [\rho_T(t_i, t_i)\Delta t], \qquad (259$$

where $\rho_T(t_k, t_i) = \rho_T(t_i, t_i)$, since $\mathbf{C}$ is a symmetrica matrix. Note, moreover, that we can also write

$$\rho_T(t_k, t_i) = \rho_T(t_i - t_j, t_i) = \rho_T(t_j - t_i, t_i), \qquad (260$$

which is a form, as we noted in Section V-A, that has particularly useful physical interpretation. Consequently the relation $\psi_N\mathbf{C}\mathbf{V} = \mathbf{Z}$ becomes

$$Z_i = \psi_N\sum_{k=1}^n C_{ik}V_k = \sum_{k=1}^n \Delta t\rho_T(t_i - t_k, t_i)V_k \qquad (261a$$

and in the limit $n \to \infty$, $\Delta t \to dt$, we get formally

$$Z(t) = \int_0^T V(t')\rho_T(t - t', t)\,dt', \quad (0 \leq t \leq T) \quad (261b$$

where we have assumed that ρ_T and V possess suitabl properties of continuity in the region $(0 \leq t, t' \leq T)$ Insertion of (261b) into (259) then yields

$$\Phi_T = \frac{2}{W_{0N}}\iint_0^T V(t_1)\rho_T(t_1, t_2)V(t_2)\,dt_1\,dt_2$$

$$= \frac{2}{W_{0N}}\int_0^T V(t_1)\,dt_1\int_0^T V(t_2)\rho_T(t_1 - t_2, t_1)\,dt_2 \qquad (262$$

for the structure factor when continuous sampling is used

Our next step is to find the integral equations whic govern ρ_T and $Z(t)$ in (261). Starting with ρ_T first, let u multiply both sides of the relation for $\mathbf{C}$ by $\psi_N(\mathbf{\Lambda}_N + \mathbf{K}_S$ and observe that the ikth element of $\psi_N\mathbf{C}(\mathbf{K}_S + \mathbf{\Lambda}_N) = \psi_N\mathbf{\Lambda}_N^{-1}\mathbf{K}_S$ is

$$\sum_i^n \psi_N C_{ij}\left[(\mathbf{K}_S)_{jk} + \frac{W_{0N}}{2}\frac{n}{T}\delta_{jk}\right] = (\mathbf{K}_S)_{ik}. \qquad (263$$

Using (261a) and passing to the limit, we get directly

$$\int_0^T \left[K_S(t, u) + \frac{W_{0N}}{2}\delta(u - t)\right]\rho_T(\tau, u)\,du$$

$$= K_S(t, \tau), \qquad (0 \leq t, \tau \leq T), \qquad (264$$

which is the desired integral equation for ρ_T, and is special case of a somewhat more general expressio

otained by Price in his treatment of the reception of atter-path noise.[43]

With the help of (261b) it is now a simple matter to erive the integral equation for $Z(t)$. To do this, we ultiply both members of (264) by $V(\tau)$ and integrate ver τ, to get

$$\int_0^T K_S(t, u)\left(\int_0^T \rho(\tau, u) V(\tau) \, d\tau\right) du$$

$$+ \frac{W_{0N}}{2} \int_0^T \rho(\tau, t) V(\tau) \, d\tau = \int_0^T V(\tau) K_S(t, \tau) \, d\tau. \quad (265)$$

sing (260) and (261b) this becomes at once

$$\int_0^T K_S(t, u) Z(u) \, du + \frac{W_{0N}}{2} Z(t)$$

$$= \int_0^T V(\tau) K_S(t, \tau) \, d\tau, \qquad (0 \le t \le T). \quad (266)$$

or stationary processes we see now that (266) and 214) are identical, provided

$$Z(t) = \frac{W_{0N}}{2} z_T(t), \qquad (0 \le t \le T), \quad (267)$$

nd so our solutions for $z_T(t)$ in Appendix IV-A, except or a scale factor, are also the solutions for $Z(t)$, under he present assumption of rational spectra and white ackground noise.

The solution of the integral equation for ρ_T may be ob- ained by direct application of the method described in etail above for $z_T(t)$. However, since ρ_T and z_T (or Z) are nearly related by (261b), we can obtain the desired rela- ions directly by comparing (231b) and (261b). The esult in the present case is, for $(0 \le t, t' \le T)$,

$$_T(t, t') = \sum_{n=1}^N h_n e^{-c_n |t-t'|} + 2 \sum_{n=1}^N E_n^{(-)}(t') e_n^{(N)} e^{-c_n(t-T)}$$

$$+ 2 \sum_{n=1}^N E_n^{(-)}(t') e_n^{(N)} e^{-c_n t} \quad (268)$$

here

$$_n^{(+)}(t') = F_n \cdot \left\{ \frac{(b_n + c_n) e^{c_n t'} - (b_n - c_n) e^{-c_n t'}}{(b_n + c_n)^2 e^{-c_n T} - (b_n + c_n)^2 e^{c_n T}} \right\}, \quad (269a)$$

$$_n^{(-)}(t') = F_n \cdot \left\{ \frac{e^{c_n(T-t')}(b_n + c_n) - e^{-c_n(T-t')}(b_n - c_n)}{(b_n - c_n)^2 e^{-c_n T} - (b_n + c_n)^2 e^{c_n T}} \right\}.$$

$$(269b)$$

rom this it is clear that we can write $\rho_T(t, t') = \rho_T(t - , t) = \rho_T(t' - t, t')$ by suitable adjustment of the ex- onential terms. For the (semi-) infinite observation imes $(T \to \infty)$, the corresponding Wiener-Hopf solution

$$_\infty(t, t') = \sum_{n=1}^N h_n \left\{ e^{-c_n|t-t'|} + \left(\frac{b_n - c_n}{b_n + c_n}\right) e^{-2c_n t} e^{c_n(t-t')} \right\}$$

$$= \rho_\infty(t - t', t), \qquad (t, t' > 0). \quad (270)$$

Eqs. (268), (270) may be specialized immediately to the RC and LRC spectra of Appendix IV-C and IV-D.

Finally, we observe that the alternative treatment of narrow-band signals in white noise (Section VII) introduces no serious modifications of the analysis: one simply replaces $V(t)$ by $V_c(t)$, $V_s(t)$ respectively, and K_S with $(K_S)_0$, in (214). For the high-Q LRC case, $(K_S)_0$ is in form identical with $(K_S)_{RC}$, so that the results of Appendix IV-C above may be used directly, with V_c, V_s, to give the corresponding $z_T(t)_c$, $z_T(t)_s$. For ρ_T in the alternative treatment, the only modification is the replacement of K_S by $(K_S)_0$.

Appendix V

A. Matched Filters

As defined above in Appendix IV-E, the *matched filter* is a linear time-invariant operator, preceding a zero-memory nonlinear element, such that the average cost of decision is minimized. Here we shall outline briefly the argument by which such matched, linear, predetection filters may be found, without, however, at- tempting to give explicit solutions in the present study.

As before, we begin with a matrix formulation and let $\mathbf{Q}$ be an $(n \times n)$ matrix (with an inverse), *i.e.*, $\det \mathbf{Q} \ne 0$, representing the matched-filter operation in the discrete case. Then

$$\mathbf{V}_F = \mathbf{Q}\mathbf{V} \quad (271)$$

is the filtered wave, when $\mathbf{V}$ is the input data. The quad- ratic form Φ, (257), characteristic of the present class of noise-in-noise problems, can now be written

$$\Phi_n = \psi_N^{-1} \tilde{\mathbf{V}} \mathbf{C} \psi_N \mathbf{V} = \psi_N^{-1} (\tilde{\mathbf{V}} \mathbf{Q})[\tilde{\mathbf{Q}}^{-1} \mathbf{C} \psi_N \mathbf{Q}^{-1}] \mathbf{Q}\mathbf{V}$$

$$= \psi_N^{-1} \tilde{\mathbf{V}}_F \mathbf{A} \mathbf{V}_F, \quad (272)$$

where $\mathbf{A} = \tilde{\mathbf{Q}} \psi_N^{-1} \mathbf{C} \mathbf{Q}^{-1}$ is dimensionless, and $\mathbf{C}$ is assumed to be symmetric. We remark that, for the moment, $\mathbf{C}$ need not be determined by the condition for optimum reception, but can represent any suboptimum receiver structure of the above quadratic type.

Our constraint of a zero-memory, nonlinear operation following the matched filter (in fact, part of the definition of matched filtering) means that the receiver structure becomes

$$\Phi_n = \psi_N^{-1} \tilde{\mathbf{V}}_F \mathbf{V}_F, \quad (273)$$

and this in turn requires that $\mathbf{A}$ be the unit matrix $\mathbf{I}$, *i.e.*,

$$\tilde{\mathbf{Q}}^{-1} \mathbf{C} \psi_N \mathbf{Q}^{-1} = \mathbf{I}; \quad (274)$$

thus, $\mathbf{C}$ is diagonalized by a congruent transformation. Since $\mathbf{C}$ is symmetric, this is certainly possible, although the diagonalization is not necessarily unique.[44]

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[43] See Bibliography [4], (54).

[44] See, for example, Bibliography [12], ch. (10).

Inasmuch as $\mathbf{Q}$ is a (discrete) linear, time-invariant filtering operation, it will have the form

$$\mathbf{Q} = [h(t_i - t_j)\Delta t], \qquad t_i = \frac{iT}{n}, \quad \text{etc.}$$

$$(i, j = 1, \cdots, n), \qquad (275)$$

where h is the weighting function of this filter. At this point we distinguish between physically realizable and nonrealizable operators: for the former the added condition that $h(t)$ vanish, all $t < 0$, is imposed, while for the latter, $h \neq 0$, $t < 0$, in general. Thus, if $\mathbf{Q}$ is to represent a physically realizable, linear filtering operation, we have

$$\mathbf{Q}_R = [h(t_j - t_i)_R \Delta t]; \qquad h_R = 0,$$

$$t_j < t_i (i, j = 1, \cdots, n). \qquad (276)$$

The general relation determining $\mathbf{Q}$ in either case is found from (274), *viz.*,

$$\psi_N \mathbf{C} = \tilde{\mathbf{Q}} \mathbf{Q}, \qquad (277)$$

which are a set of simultaneous, *nonlinear* equations for the Q_{ij}.

In the discrete case, (274) gives $\psi_N \mathbf{C} = \tilde{\mathbf{Q}}_R \mathbf{Q}_R$, and with $\psi_N \mathbf{C} = [\rho_T(t_i, t_k)_R \Delta t]$, cf. (260), this relation becomes specifically

$$\rho_T(t_k, t_k)_R = \sum_i h(t_i - t_i)_R h(t_i - t_k)_R \Delta t,$$

$$(i, k = 1, \cdots, n). \qquad (278)$$

The continuous operations are of chief interest. Passing to the limit, we get from (278) the basic *nonlinear* integral equation

$$\rho_T(t, \tau)_R = \int_0^T h(x - t)_R h(x - \tau)_R \, dx,$$

$$(0 \leq t, \tau \leq T), \qquad (279)$$

corresponding to (278), where it is assumed that $(\rho_T)_R$ is given. For the Bayes, or optimum system discussed in the text, $\mathbf{C}$ has the specific form (244), and $(\rho_T)_R$ is then the solution of (265), so that $h_R \rightarrow (h_R)_{\text{opt.}}$ in (279). The condition of physical realizability, of course, means that $h(x - t)_R = 0$, $x - t < 0$, etc. [This is reflected in the fact that we leave the receiver's structure Φ_T unaltered, if we set $\rho_T = 0$ for $t - \tau < 0$, (see Section V-A.] Observe from (273), with $\psi_N^{-1} = (2/W_0)T/n$, that in the limit $(n \rightarrow \infty)$, or $T/n_i = \Delta t \rightarrow dt$, we get

$$\Phi_T = \frac{2}{W_{0N}} \int_0^T V_F(t)_R^2 \, dt, \qquad (280)$$

with

$$V_F(t)_R = \int_0^T V(t')h(t - t_R') \, dt', \qquad (0 \leq t \leq T), \quad (281)$$

from (271), (275), for continuous sampling. Inserting (281) into (280) and using (279), we have alternatively

$$\Phi_T = \frac{2}{W_{0N}} \iint_0 V(t_1)\rho_T(t_1, t_2)_R V(t_2) \, dt_1 \, dt_2, \qquad (282)$$

and in the case of optimum systems, this is identical with (263), where ρ_T is obtained from (265), and $h_R \rightarrow (h_R)_{\text{opt.}}$.

For the $\mathbf{C}$'s or ρ_T's, that have their counterparts in a physical process (implying suitable continuity properties on ρ and h), we expect that (279) possesses a meaningful solution, since the diagonalization of the symmetric matrix $\mathbf{C}$ from which (279) follows in the limit is always possible with a congruent transformation. The reverse situation, where $h(t)_R$ is specified beforehand as one condition of the problem, presents no difficulties: the nonlinear device is still square-law with zero memory, but the receiver structure is no longer in general optimum, since ρ_T does not then correspond to the proper weighting $\mathbf{C} = \mathbf{\Lambda}_N^{-1} - (\mathbf{\Lambda}_N + \mathbf{K}_S)^{-1}$ (for white noise backgrounds) of the optimum decision system in the discrete case, nor is ρ_T a solution of the corresponding (265) for continuous operation in $(0, T)$.

The solution of our fundamental nonlinear (279) can be achieved as follows: we observe first that $\rho_T(t, \tau)_R$ is required to vanish outside the square $(0 \leq t, \tau \leq T)$, from (70), since only operations on data in $(0, T)$ can influence the (binary) decision process; $(V(t)$ is assumed to be available for all $-\infty < t < \infty$). Thus, we have

$$\left.\begin{aligned}\rho_T(t, \tau)_R &= \text{solution of (70)}, \qquad (0 \leq t, \tau \leq T) \\ &= 0, \qquad t, \tau \text{ outside} \quad (0 \leq t, \tau \leq T)\end{aligned}\right\}. \quad (283)$$

Taking the double Fourier transform of both members of (279) then yields directly

$$P_T(i\omega) \equiv \iint_0^T \rho_T(t, \tau)_R e^{-i\omega(t-\tau)} \, dt \, d\tau = T \mid Y(i\omega)_R \mid^2, \quad (284)$$

from which it follows at once that

$$\mid Y(i\omega)_R \mid = T^{-1/2}\left\{\iint_0^T \rho_T(t, \tau)_R e^{-i\omega(t-\tau)} \, dt \, d\tau\right\}^{1/2}. \quad (285)$$

That the double integral is a real quantity (ρ_T real, of course), may be alternatively demonstrated with the help of Mercer's theorem,[45] if we note from (260) that ρ_T is symmetrical in its arguments. Furthermore, since $\rho_T(t, \tau)$ is assumed positive definite,[45] we can expand it in the bilinear form (Mercer's theorem)

$$\rho_T(t, \tau)_R = \sum_{n=0}^{\infty} \lambda_n \phi_n(t) \phi_n(\tau) \qquad (286)$$

where the λ_n are all consequently real and positive (some may be zero). These λ_n are, of course, the eigenvalues of the integral equation

$$\int_0^T \rho_T(t, \tau)_R \phi_n(\tau) \, d\tau = \lambda_n \phi_n(t); \qquad (0 \leq t \leq T). \quad (287)$$

Taking the double Fourier transform again, with (286) for $\rho_T(t, \tau)_R$, yields

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[45] Bibliography [6], (16). From physical considerations (our system is an *energy* detector, essentially), it is evident that Φ_T, (73) can never be negative.

$$P_T(i\omega) = \sum_{n=0}^{\infty} \lambda_n \int_0^T \phi_n(\tau)e^{i\omega\tau}\, d\tau \cdot \int_0^T \phi_n(t)e^{-i\omega t}\, dt$$

$$= \sum_{n=0}^{\infty} \lambda_n \mid a_n(i\omega) \mid^2 > 0, \qquad (288)$$

with $a_n(i\omega) = \int_0^T \phi_n(t) \exp{(-i\omega t)}dt$, and this immediately establishes the positive, real nature of $P_T(i\omega)$. Thus, in place of the exponent in (285) we equally well use os $\omega(t - \tau)$.

ACKNOWLEDGMENT

The author is indebted to Dr. Robert Price, Lincoln Laboratory, Massachusetts Institute of Technology, for numerous stimulating discussions of this topic and his related work, including a careful reading of the present material. The author also wishes to thank Prof. James E. Storer for some illuminating discussions of the integral equation.

BIBLIOGRAPHY

[1] Middleton, D. and Van Meter, D. "Detection and Extraction of Signals in Noise from the Point of View of Statistical Decision Theory, Parts I and II," *Society of Industrial and Applied Mathematics*, Vol. 3 (December, 1955), and Vol. 4 (June, 1956), pp. 192, 86.
[2] Davis, R. C. "Detectability of Random Signals in the Presence of Noise," IRE TRANSACTIONS, PGIT-3 (March, 1954), pp. 52-62.
[3] Bussgang, J. J. and Middleton, D. "Optimum Sequential Detection of Signals in Noise," IRE TRANSACTIONS, Vol. IT-1 (December, 1955), pp. 5-18.
[4] Price, R. "Optimum Detection of Random Signals in Noise, with Applications to Scatter-Multipath Communication, Part I," IRE TRANSACTIONS, Vol. IT-2 (December, 1956), pp. 125-135. Part II to be published in the TRANSACTIONS at a later date. Parts I and II based on notes of R. Price, issued as group reports 34-39, and 40, May, 1955, October, 1955, of Group 34, Lincoln Laboratory, Massachusetts Institute of Technology. (These group reports are not generally available.)
[5] Kac, M. and Siegert, A. J. F. "On the Theory of Noise in Radio Receivers with Square-Law Detectors," *Journal of Applied Physics*, Vol. 18 (April, 1947), pp. 383-397.
[6] Emerson, R. C. "First Probability Densities for Receivers with Square-Law Detectors," *Journal of Applied Physics*, Vol. 24 (September, 1953), pp. 1168-1176.
[7] Cramér, H. *Mathematical Methods of Statistics*, Princeton, Princeton University Press, 1946.
[8] Van Vleck, J. H. and Middleton, D. "A Theoretical Comparison of the Visual, Aural, and Meter Reception of Pulsed Signals in the Presence of Noise," *Journal of Applied Physics*, Vol. 17 (November, 1946), pp. 940-971.
[9] Van Meter, D. and Middleton, D. "Modern Statistical Approaches to Reception in Communication Theory," IRE TRANSACTIONS, PGIT-4 (September, 1954), pp. 119-145.
[10] Peterson, W. W., Birdsall, T. G., and Fox, W. C. "The Theory of Signal Detectability," IRE TRANSACTIONS, PGIT-4 (September, 1954), pp. 171-212.
[11] Pearson, K. *Tables of the Incomplete Function*, Cambridge, England, Cambridge University Press, 1951.
[12] Margenau, H. and Murphy, G. M. *The Mathematics of Physics and Chemistry*, New York, D. Van Nostrand Company, Inc., 1943.
[13] Lovitt, W. V. *Linear Integral Equations*, New York, Dover Publications, Inc., 1950, p. 32.
[14] Meyer, M. and Middleton, D. "On the Distribution of Signals and Noise after Rectification and Filtering," *Journal of Applied Physics*, Vol. 25 (August, 1954), pp. 1037-1052.
[15] Juncosa, M. L. "An Integral Equation Related to Bessel Functions," *Duke Mathematical Journal*, Vol. 12 (1945), pp. 465-471.
[16] *Tables of the Error Function and Its First Twenty Derivations*, Cambridge, Harvard University Press, 1951.
[17] Zadeh, L. A. and Ragazzini, J. R. "Optimum Filters for the Detection of Signals in Noise," PROCEEDINGS OF THE IRE, Vol. 40 (October, 1952), pp. 1223-1231.
[18] Middleton, D. Lincoln Laboratory Report 35, Massachusetts Institute of Technology (November, 1953), (not generally available).
[19] Siegert, A. J. F. "A Systematic Approach to a Class of Problems in the Theory of Noise and Other Random Phenomena, Part II, Examples," P-730 The RAND Corporation, Santa Monica, California (September, 1955). Also, IRE TRANSACTIONS, Vol. IT-3 (March, 1957), pp. 38-43.
[20] Turin, G. "Communication Through Noisy, Random-Multipath Channels," 1956 IRE CONVENTION RECORD, Part 4, pp. 154-166. Subsequent results to be published in a later date of TRANSACTIONS.
[21] Whittle, P. *Hypothesis Testing in Time Series Analysis*, Almquist and Wiksells A. B., Uppsala, Sweden, 1951, p. 13.

The Relationship of Sequential Filter Theory to Information Theory and Its Application to the Detection of Signals in Noise by Bernoulli Trials*

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Summary—In this paper the problem of detecting signals in noise by the method of sequential filtering is formulated. A slicing operator for converting a given random variable into a Bernoulli random variable is defined. A method for choosing an optimum slicing operator in a certain prescribed sense is given. It is also shown that the Bernoulli sequential test is defined by three parameters a (p_0, p_1, α, β), b(p_0, p_1, α, β), and c(p_0, p_1) rather than four, as one would normally expect. The significance of these transformations is briefly discussed. Finally, the theory of Bernoulli sequential detection is applied to the detection of a sine-wave carrier in noise when the signal-to-noise ratio is less than one. The efficiency of this detector is calculated and compared with the results of others. Curves of the significant Bernoulli sequential detector characteristics are given for this problem.

I. Formulation of the Detection Problem

LET E_0 and E_1 be two mutually exclusive events whose occurrence an observer wants to detect. Hence, if the event E_0 occurs, then E_1 cannot logically occur, and if E_1 occurs, then E_0 cannot occur. It is known, however, that the probability is unity that one of these will be present when the observation is started. The observer knows *a priori* that the event E_0 produces one of a possible number of effects $f_0(t)$ which is a member of the set S_0, and E_1 produces one of a possible number of effects $f_1(t)$ which is a member of the set S_1. We assume that the sets S_0 and S_1 are disjoint. In the problem considered it is assumed that the information functions $f_0(t)$ and $f_1(t)$ can take on only two values, "zero" or "one," during different time intervals. This can be realized by a slicing operation. These functions are sampled so that the zeros and ones which result are statistically independent. If p is the probability of a "one" when a member of S_0 or S_1 is observed, then S_0 contains all $f_0(t)$ for which $p \leq p_0$, and the set S_1 contains all $f_1(t)$ for which $p \geq p_1$ such that $p_1 > p_0$. If the results of an observation indicate that $p \leq p_0$, then the observed information function belongs to S_0 and the hypothesis H_0 is accepted that the event E_0 occurred. Similarly, if the results indicate that $p \geq p_1$, then the observed information function belongs to S_1 and the hypothesis H_1 is accepted (or H_0 is rejected) that E_1 occurred.

Since this is a two valued decision problem, there are two types of errors possible. For example, it is possible to accept H_1 when, in fact, H_0 is true or to accept H_0 when, in fact, H_1 is true. Let α be the probability of accepting H_1 (signal-plus-noise is present) when H_0 (only noise is present) is true and $1 - \alpha$ is the probability of accepting H_0 when H_0 is true. Also β is the probability of accepting H_0 (only noise is present) when, in fact, H_1 (signal-plus-noise is present) is true and $1 - \beta$ is the probability of accepting H_1 when H_1 is true.

In sequential detection, the problem considered here, the observer specifies the error probabilities α, β along with the threshold parameters p_0 and p_1. A criterion must now be established for choosing an optimum procedure.

II. Brief Introduction to the Theory of Sequential Detection

In this section the highlights of sequential detection theory will be outlined. The equations presented here will then be specialized to Bernoulli detection. For the mathematical details the reader is referred to the literature [1, 12].

A sequential test can be described as follows: at $t = t_1$, the observer measures the value x_1. Based on this value it must be decided whether to accept the hypothesis H_0 that the parameter of the distribution $P(x, \theta)$ is $\theta \leq \theta_0$, or H_1 that $\theta \geq \theta_1$, or whether the datum is insufficient to accept either one of the hypotheses with confidence. (For the Bernoulli case we will use the symbol p instead of θ.) If H_0 or H_1 is accepted, experimentation is terminated. On the other hand, if the single datum is insufficient to lead to the acceptance of one of these hypotheses, then at $t = t_2$, the observer takes another sample x_2. Based on the sample of size two (x_1, x_2) the observer must once again make one of three possible decisions: accept H_0 or H_1, or the datum is insufficient for accepting either one. If H_0 or H_1 is accepted, the experiment is terminated. If the data are insufficient for a terminating decision, x_3 is observed. The same decision procedure is then repeated on the sample point (x_1, x_2, x_3). Experimentation is continued in this manner until either H_0 or H_1 is accepted. The number of samples required for the termination of a sequential test is a random variable. From the class of all sequential experiments of strength α, β and threshold parameters θ_0 and θ_1 the decision rule which minimizes the average number of samples required for termination at these parameters is chosen. The optimum decision

procedure for independent sampling requires the acceptance of the hypothesis H_0 at that value of n for which,

$$z(n) = \sum_{i=1}^{n} \log \frac{P(x_i; \theta_1)}{P(x_i; \theta_0)} \leq \log \frac{\beta}{1 - \alpha} \qquad (1)$$

and H_1 is accepted at that value of n for which,

$$z(n) = \sum_{i=1}^{n} \log \frac{P(x_i; \theta_1)}{P(x_i; \theta_0)} \geq \log \frac{1 - \beta}{\alpha} \qquad (2)$$

where,

$P(x, \theta)$ = probability density function of the random variable x when θ is the true parameter,

α = probability of accepting the hypothesis H_1 that $\theta \geq \theta_1$ when, in fact, H_0 or $\theta \leq \theta_0$ is true,

β = probability of accepting the hypothesis H_0 that $\theta \leq \theta_0$ when, in fact, H_1 or $\theta \geq \theta_1$ is true,

$1 - \alpha$ = probability of accepting H_0 when H_0 is, in fact, true,

$1 - \beta$ = probability of accepting H_1 when H_1 is, in fact, true.

Also, θ_1 is always understood to be greater than θ_0.

In a filtering process there exist certain mathematical functions which are used as criteria for judging the performance of the filter. There are two such primary characteristics in sequential detection theory, although others can be used to supplement these. The two most important characteristics for judging sequential detection performance are the operating characteristic function called the OC function, and the average sample number function, called the ASN function.

The OC function $L(\theta)$, gives the conditional probability of accepting H_0 when θ is the true parameter of the distribution. It is a strictly decreasing function of θ which takes on the values $1 - \alpha$ at the threshold parameter $\theta = \theta_0$ and β at the threshold parameter $\theta = \theta_1$. This function gives the confidence with which H_0 is accepted as a function of the parameter θ. For independent sampling, the mathematical expression is given to a very good approximation by the set of parametric equations,

$$E_\theta[e^{zh}] = 1, \qquad (3)$$

and

$$L(h) = \frac{\left(\frac{1 - \beta}{\alpha}\right)^h - 1}{\left(\frac{1 - \beta}{\alpha}\right)^h - \left(\frac{\beta}{1 - \alpha}\right)^h} ; \qquad -\infty \leq h \leq \infty \qquad (4)$$

$$z = \log \frac{P(x; \theta_1)}{P(x; \theta_0)}. \qquad (5)$$

At the point $h = 0$ which corresponds to the point $\theta = \theta'$ at which $E_\theta(z) = 0$, $L(h)$ is indeterminate. By an application of L'Hospital's Rule or some equivalent it is found that,

$$L(\theta') = \frac{\log \left(\frac{1 - \beta}{\alpha}\right)}{\log \left(\frac{1 - \beta}{\alpha}\right) + \log \left(\frac{1 - \alpha}{\beta}\right)} ; \quad \theta_0 < \theta' < \theta_1. \qquad (6)$$

Eq. (3) gives the expected value of e^{zh} on condition θ. The function $L(h)$ is a universal curve for all sequential tests of strength α, β. Eq. (3) relates this curve to the statistics of the random variable being measured. For a given value of h, a corresponding value of θ is obtained from (3). For the same value of h, a corresponding value $L(h)$ is obtained from (4). This gives the point $[L(\theta), \theta]$ on the OC function. The procedure is repeated for all h. At the threshold parameter θ_0, $h = 1$ and at θ_1, $h = -1$. The OC function given by (3) and (4) is not exact, since in the development it was assumed that experimentation always terminates when the equality signs in (1) and (2) are satisfied exactly. This, however, is not true for discrete sampling. The excess over the boundaries is neglected which introduces errors that are of no practical importance. The author has verified this experimentally [4].

The second important characteristic, the ASN function, gives the average number of samples required for a sequential test to terminate when θ is the true parameter. The mathematical expression for the ASN function is given by

$$E_\theta(n) = \frac{L(\theta) \log \frac{\beta}{1 - \alpha} + [1 - L(\theta)] \log \frac{1 - \beta}{\alpha}}{E_\theta(z)} \qquad (7)$$

where,

$E_\theta(n)$ = average number of samples required for a sequential test to terminate when θ is the true parameter,

$E_\theta(z)$ = average value of the random variable which is measured on condition that θ is the true parameter. [z is defined in (5).]

At the indeterminate point $\theta = \theta'$, the ASN function takes on (for all practical purposes) its maximum value given by

$$E_{\theta'}(n) = \frac{\left(\log \frac{1 - \beta}{\alpha}\right)\left(\log \frac{1 - \alpha}{\beta}\right)}{E_{\theta'}(z^2)} ; \quad \theta_0 < \theta' < \theta_1. \qquad (8)$$

Since $L(\theta)$ is an approximation, it follows that $E_\theta(n)$ is only approximate. Once again the error introduced by neglecting the excess over the boundaries upon termination of the experiment is generally of no practical importance. At the threshold parameters θ_0, $L(\theta_0) = 1 - \alpha$ hence

$$E_{\theta_0}(n) = \frac{(1 - \alpha) \log \frac{\beta}{1 - \alpha} + \alpha \log \frac{1 - \beta}{\alpha}}{E_{\theta_0}(z)}. \qquad (9)$$

Similarly, at $\theta = \theta_1$, $L(\theta_1) = \beta$ and,

$$E_{\theta_1}(n) = \frac{\beta \log \frac{\beta}{1 - \alpha} + (1 - \beta) \log \frac{1 - \beta}{\alpha}}{E_{\theta_1}(z)}. \qquad (10)$$

We define the function $E_\theta(z)$ in (7) as the "average information per sample gained from measurement" when the data come from a distribution whose parameter is θ.

The properties of (11) as a measure of information, have been studied in detail [6, 8, 9]. For given α, β, the average number of samples at θ_0 and θ_1 decreases with increase in the average information per sample gained from measurement. Eq. (9) and (10) relate in a very simple and compact manner the three fundamental properties which characterize a two value decision problem; the average number of samples, the (α, β) errors, and the average information per sample gained from measurement.

The average amount of information from observation required for a sequential test to terminate when θ_0 is true for specified error probabilities α, β is given by the numerator of (9). The numerator of (10) gives the average information required for termination when θ_1 is the true parameter for specified α, β.

The function $E_\theta(z)$ will play a very important part in optimizing a Bernoulli detector. The mathematical expression for $E_\theta(z)$ is given by

$$E_\theta(z) = \int_{-\infty}^{\infty} P(x; \theta) \log \frac{P(x; \theta_1)}{P(x; \theta_0)} \, dx. \qquad (11)$$

III. Bernoulli Sequential Detection Characteristics

A. Introduction

In many problems of interest, the optimum detector characteristic required by sequential filter theory cannot be realized either due to the fact that the statistics of the signal are not completely known *a priori* or due to certain practical constraints imposed upon the observation. Under such circumstances, it would still be desirable to synthesize a sequential filter whose performance is sufficiently good when compared to the best filter given by the theory, for example, when all the *a priori* statistics are used. That is, for the same error probabilities (α, β) and threshold parameters θ_0, θ_1 we would like a filter such that the average number of samples required for termination is comparable to the optimum in the operating region of interest. One such filter is the Bernoulli sequential detector [2, 4, 12] which will now be discussed.

B. Formulation of Sequential Detection by Bernoulli Trials

Consider a sequence of independent sample values $x_1 \cdots x_n$ taken from a random variable belonging to a one parameter family of probability density functions $P(x, \theta)$. On each sample value we define the operation R such that

$$R_{x_0}[x_i] = r_i(x_0) \qquad (12)$$

where

$$r_i(x_0) = 1 \quad \text{when} \quad x_i \geq x_0$$
$$= 0 \quad \text{when} \quad x_i < x_0.$$

The operation R_{x_0} on each of the samples of the sequence generates a new sequence of independent samples, $r_1 \cdots r_n$. The samples in the new sequence can only take on the values zero or one. The probability of choosing k_n ones and $n - k_n$ zeroes from such a sample of size n is given by the familiar Bernoulli distribution

$$P_n(k_n, x_0) = \frac{n!}{k_n!(n - k_n)!} [p(x_0)]^{k_n}[1 - p(x_0)]^{n-k_n}$$

$$k_n = 0, 1, 2, \cdots, n \qquad (13)$$

where, $p(x_0) = p(x \geq x_0) =$ the probability of a "one" at the output of slicing operator R_{x_0}, or the probability that the random variable x, before slicing, exceeds x_0. It should be clear that, by varying the slicing threshold x_0, a family of Bernoulli random variables is generated and hence a family of Bernoulli distributions whose parameter p depends on the threshold x_0.

Let us now assume that the input to the operator R_{x_0} is either noise, whose probability density function is $P(x; \theta_0)$, or signal-plus-noise whose density is $P(x; \theta_1)$. The cumulative distribution of x on condition θ is given by

$$p(x_0, \theta) = \int_{x_0}^{\infty} P(x, \theta) \, dx. \qquad (14)$$

For a given value of θ, $p(x_0, \theta)$ is a function of the slicing threshold x_0. For a given value of x_0, $p(x_0, \theta)$ depends on θ. Without introducing confusion, for convenience, we replace $p(x_0, \theta)$ by $p(x, \theta)$ where x is an arbitrary slicing threshold. We assume that $p(x, \theta)$ belongs to the class of one parameter family of distributions such that for any given slicing threshold x, $p(x, \theta)$ is strictly monotonic in θ. This includes the well known Gaussian and Rayleigh families as well as others that are of physical interest. Corresponding to a test of hypothesis $\theta \leq \theta_0$ against $\theta \geq \theta_1$, we obtain an equivalent Bernoulli test

$$p(x, \theta) \leq p(x, \theta_0) = p_0,$$
$$p(x, \theta) \geq p(x, \theta_1) = p_1.$$
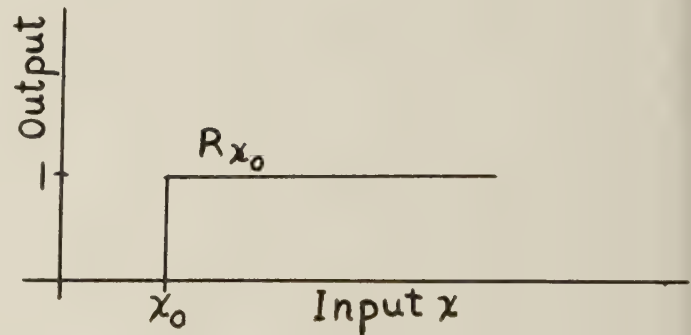
Fig. 1—Slicing operator R_{x_0}; $R_{x_0}[x] = 1$, $x \geq x_0$
$= 0$, $x < x_0$

Fig. 1 illustrates the slicing characteristic of the operator R_{x_0}. The operator converts any input function into a two valued output function.

Fig. 2 illustrates the cumulative distribution function $p(x, \theta)$ of the random variable x for two parameters θ_0, θ_1 of the probability density $p(x, \theta)$. The figure illustrates the properties of the class of distributions considered. For any slicing threshold x_0, $p(x_0, \theta) = p(x \geq x_0; \theta)$
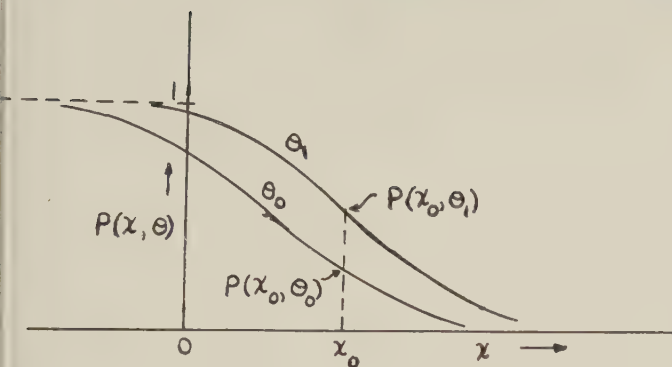
Fig. 2—Probability distribution of random variable x for parameters θ_0, θ_1, $(\theta_1 > \theta_0)$.

ncreases with increase in the parameter θ. As an example, f θ_0 and θ_1 represent the signal-to-noise ratio of a signal -mbedded in noise, then Fig. 2 shows that the greater the ignal-to-noise ratio the greater the probability that the andom variable x will exceed the threshold x_0. This is rue for any x_0.

Fig. 3—Probability distributions of random variable x for parameters θ_0, θ_1 after slicing by operator R_{x_0}.
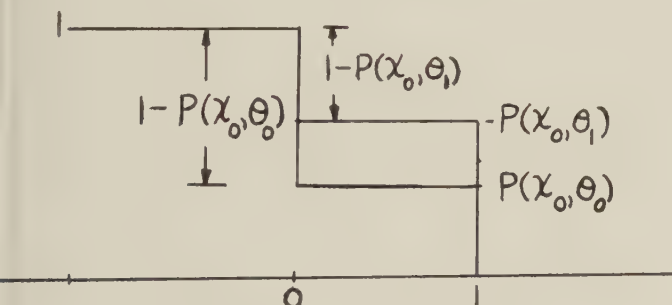
Fig. 3 is the distribution function of the random variable x after being sliced by the operator R_{x_0} of Fig. 1. Since the operator converts x into a random variable which takes on the values zero and one, the two valued variable has a discontinuous distribution function as shown in Fig. 3. It takes on the value zero with probability $1 - p(x_0, \theta)$ and the value unity with probability $p(x_0, \theta)$. This is equivalent to saying that before slicing $p(x_0, \theta)$, is the probability that x is greater than or equal to x_0, while . $- p(x_0, \theta)$ is the probability that $x < x_0$. The probability of obtaining k_n ones and $n - k_n$ zeroes in n independent amples is given by the convolution of n distributions of he form shown in Fig. 2. This, of course, leads to the vell-known Bernoulli distribution.

C. Bernoulli Detection Characteristics

Statistical Decision Regions: From (13) the logarithm f the probability ratio for the Bernoulli detector is given by

$$(n, x) = k_n\left[\log\frac{p_1}{p_0} - \log\frac{1 - p_1}{1 - p_0}\right] + n\log\frac{1 - p_1}{1 - p_0} \quad (15)$$

vhere for given values of θ_0 and θ_1, p_0 and p_1 are functions f x. As previously shown, the Bernoulli random variable f probability ratio $z(n, x)$ is obtained by slicing the

random variable of density $p(x, \theta)$ at the threshold x. The optimum decision regions are obtained from (1), (2), and (15) as

$$z(n, x) \leq \log\frac{\beta}{1 - \alpha}; \quad \text{accept} \quad H_0 \quad (16a)$$

$$z(n, x) \geq \log\frac{1 - \beta}{\alpha}; \quad \text{accept} \quad H_1 \quad (16b)$$

$$\alpha, \beta < \tfrac{1}{2}.$$

For any value of n for which one of the inequalities is satisfied first, the corresponding hypothesis is accepted. Inequalities (16) in conjunction with (15) can be expressed as

$$k_n \leq \frac{\log\dfrac{\beta}{1 - \alpha}}{\log\dfrac{p_1}{p_0} + \log\dfrac{1 - p_0}{1 - p_1}}$$

$$+ n\frac{\log\dfrac{1 - p_0}{1 - p_1}}{\log\dfrac{p_1}{p_0} + \log\dfrac{1 - p_0}{1 - p_1}} \to H_0 \quad (17)$$

$$k_n \geq \frac{\log\dfrac{1 - \beta}{\alpha}}{\log\dfrac{p_1}{p_0} + \log\dfrac{1 - p_0}{1 - p_1}}$$

$$+ n\frac{\log\dfrac{1 - p_0}{1 - p_1}}{\log\dfrac{p_1}{p_0} + \log\dfrac{1 - p_0}{1 - p_1}} \to H_1. \quad (18)$$

The arrow indicates the hypothesis to which the acceptance region belongs.

Let us now define the following three transformations:

$$a = \frac{\log\dfrac{1 - \beta}{\alpha}}{\log\dfrac{1 - p_0}{1 - p_1}}, \quad (19)$$

$$b = \frac{\log\dfrac{\beta}{1 - \alpha}}{\log\dfrac{1 - p_0}{1 - p_1}}, \quad (20)$$

$$c = \frac{\log\dfrac{p_1}{p_0}}{\log\dfrac{1 - p_0}{1 - p_1}}. \quad (21)$$

Substituting these into (17) and (18) yields the following decision regions

$$k_n \leq \frac{b}{c + 1} + \frac{n}{c + 1} \to H_0 \quad (22)$$

$$k_n \geq \frac{a}{c + 1} + \frac{n}{c + 1} \to H_1. \quad (23)$$

It is seen that the a, b, c transformations completely define the sequential Bernoulli detector. Hence, when the

observer specifies the parameter point $(p_0, p_1, \alpha, \beta)$, he completely specifies the a, b, c, transformations. However, an *infinity* of such parameter points exists which specify the same a, b, c transformation and hence, the same sequential test. Furthermore, since the sequential test is optimum [1] at the parameter point $(p_0, p_1, \alpha, \beta)$ and since there exists an infinity of such points which define the same test, it follows that a given sequential Bernoulli experiment as defined by the set a, b, c is optimum at an infinity of parameter points which satisfy the same a, b, c. Fig. 4 is a curve of (21), for $c = 4$ and $c = 8$. It is seen that an infinity of points (p_0, p_1) exists which lead to the same c. For a given (a, b) the corresponding values of α, β can be obtained from (19) and (20).

D. Performance Characteristics of a Bernoulli Detector

The Operating Characteristic Function (OC Function): The OC function discussed in Part II gives the probability
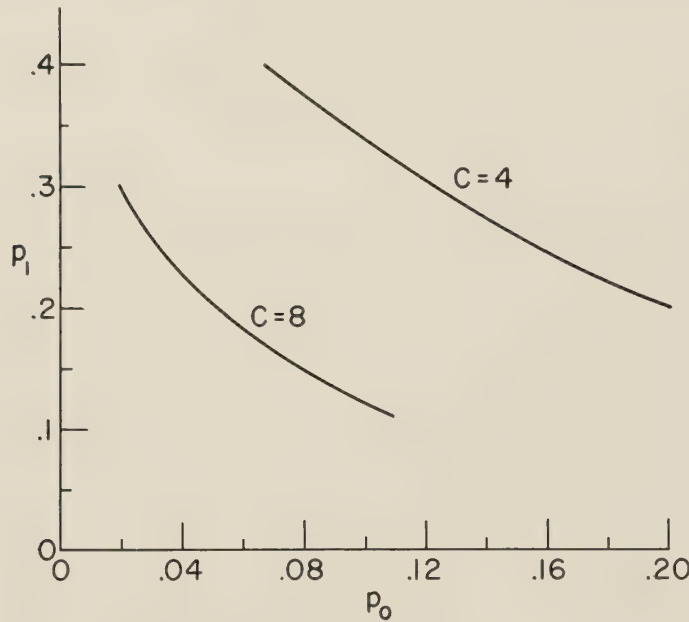

Fig. 4—Graph of p_1 vs p_0 for two values of

$$c = \frac{\log \dfrac{p_1}{p_0}}{\log \dfrac{1 - p_0}{1 - p_1}}$$

for Bernoulli sequential filter.

of accepting H_0 when p is the true parameter. Eq. (4) is the same for all sequential tests of strength α, β. Hence,

$$L_p(h) = \frac{\left(\dfrac{1 - \beta}{\alpha}\right)^h - 1}{\left(\dfrac{1 - \beta}{\alpha}\right)^h - \left(\dfrac{\beta}{1 - \alpha}\right)^h} \quad (24)$$

where $L_p(h)$ indicates that the Bernoulli case is considered. It is now necessary to apply (3) to the Bernoulli case. The random variable z takes on two values, $\log p_1/p_0$ with probability p and $\log (1 - p_1/1 - p_0)$ with probability $1 - p$. Hence, from (3), or,

$$E_p[e^{zh}] = p \exp\left\{\left[h \log \frac{p_1}{p_0}\right]\right\}$$

$$+ (1 - p) \exp\left\{\left[h \log \frac{1 - p_1}{1 - p_0}\right]\right\} = 1$$

$$E_p[e^{zh}] = p\left(\frac{p_1}{p_0}\right)^h + (1 - p)\left(\frac{1 - p_1}{1 - p_0}\right)^h = 1.$$

Solving for p, we obtain

$$p(h) = \frac{1 - \left(\dfrac{1 - p_1}{1 - p_0}\right)^h}{\left(\dfrac{p_1}{p_0}\right)^h - \left(\dfrac{1 - p_1}{1 - p_0}\right)^h}, \quad (25)$$

where $-\infty \le h \le \infty$. From (24) and (25), the OC function $L(p)$ can be obtained. If it is desired to plot $L_p(\theta)$, it is only necessary to obtain $p(\theta)$ corresponding to some slicing threshold x, where

$$p(x, \theta) = \int_x^\infty P(y, \theta) \, dy. \quad (26)$$

For given values of α, β, points on $L_p(h)$ are computed. For the same values of h, points on $p(h)$ are obtained. This results in the function $L(p)$. From (26), p is obtained as a function of θ. The OC function $L_p(\theta)$ is then obtained from $L(p)$ and $p(\theta)$. Corresponding to the parameters θ_0 and θ_1 and some slicing threshold x, we have the Bernoulli parameters $p_0 = p(x, \theta_0)$, $p_1 = p(x, \theta_1)$. At the threshold parameters θ_0, θ_1, $h(\theta_0) = 1$, $h(\theta_1) = -1$. Hence,

$$L_{p_1}(\theta_1) = L(\theta_1) = \beta,$$

and

$$L_{p_0}(\theta_0) = L(\theta_0) = 1 - \alpha.$$

It is seen that the OC function $L_p(\theta)$ of the Bernoulli test corresponding to some slicing threshold $x = x_0$ has the same value at θ_0, θ_1 as the OC function $L(\theta)$, where $L(\theta)$ corresponds to the random variable x, before slicing, whose probability density is $P(x, \theta)$.

Let us now obtain the OC function in terms of the a, b, c transformations previously defined. If we solve (19) and (20) for $\log (1 - \beta/\alpha)$ and $\log (\beta/1 - \alpha)$ and introduce the transformation

$$u = \exp\left\{h \log \left(\frac{1 - p_0}{1 - p_1}\right)\right\}, \quad (27)$$

we obtain

$$L_p(u) = \frac{u^a - 1}{u^a - u^b}. \quad (28)$$

From (27), it is clear that as h ranges over the entire real line, u ranges only over the positive half of the real line. At the point $u = 1$ (corresponding to $h = 0$), an application of L'Hospital's Rule yields

$$L(1) = \frac{a}{a - b}. \quad (29)$$

Note from (20) that $b < 0$, since $\alpha < 0.5$, $\beta < 0.5$, $p_0 < p_1$.]

It is now necessary to express (25) in terms of the u variable. This can be easily accomplished by solving (21) for p_1/p_0, substituting this result into (25), and then introducing the change of variable given by (27). This yields

$$p(u) = \frac{u - 1}{u^{c+1} - 1}. \qquad (30)$$

At the indeterminate point $u = 1$,

$$p(1) = \frac{1}{c + 1}. \qquad (31)$$

$L(p)$ can be obtained from (28) and (30). Furthermore, $p(x, \theta)$ is given by (26). The function $L_p(\theta)$ can now be obtained for some slicing threshold x.

The Average Sample Number Function (ASN Function): Up to this point we have shown that a random variable of probability density function $P(x, \theta)$ can be converted, by means of a slicing operator R_x, into a random variable that can take on two values one and zero, with probabilities $p(x, \theta)$ and $1 - p(x, \theta)$. The function $p(x, \theta)$ gives the probability that the random variable takes on a value greater than or equal to x, when θ is the true parameter. For each slicing threshold x and for some parameter θ, a two valued random variable is realized whose distribution depends on x.

We have also shown that, corresponding to a sequential test defined by the parameter point $(\theta_0, \theta_1, \alpha, \beta)$ for the random variable before slicing, there exists a Bernoulli sequential test defined by $\{p(x, \theta_0), p(x, \theta_1), \alpha, \beta\}$ for the random variable after slicing. The Bernoulli test is clearly a function of the slicing threshold x. However, at the threshold parameters θ_0, θ_1, the Bernoulli test can have the same strength α, β, as the random variable before slicing. It should be clear that the slicing operator R_x destroys information which is useful in the decision process. Thus for the same $(\theta_0, \theta_1, \alpha, \beta)$ we expect the Bernoulli test to require, on the average, more samples than the corresponding test before slicing by the operator R_x. The ASN functions at the points $(\theta_0, \theta_1, \alpha, \beta)$ can be used to compare the two tests. The ASN function of the Bernoulli test of $\{p(x, \theta_0), p(x, \theta_1), \alpha, \beta\}$ will be a function of the slicing threshold x. It is desirable to choose the threshold x or the operator R_x for a given α, β in such a way that the ASN function is a minimum at either θ_0 or θ_1, or perhaps at the point $\theta = \theta'$, $\theta_0 < \theta' < \theta_1$, where the ASN function is a maximum. It is this phase of the detection problem which will now be considered.

In order to obtain the ASN function for the Bernoulli case, it is necessary to make use of the general expression given in (7). The numerator remains the same except that, for the Bernoulli case, $L(\theta)$ is replaced by $L_p(\theta)$, where $L_p(\theta)$ is obtained by making use of (24)-(26), as outlined in the previous section on the OC function. In order to obtain $E_\theta(z)$ for the Bernoulli case, it is necessary to recall that z takes on only two values, log p_1/p_0 with probability p, and log $(1 - p_1/1 - p_0)$ with

probability $1 - p$. It is also necessary to remember that the Bernoulli parameters p_0, p_1, and p are functions of the slicing threshold x and the parameter θ. Hence, the average value of z on condition $p = p(x, \theta)$ is given by

$$E_p(x, \theta; z) = p(x, \theta) \log \frac{p(x, \theta_1)}{p(x, \theta_0)}$$
$$+ [1 - p(x, \theta)] \log \frac{1 - p(x, \theta_1)}{1 - p(x, \theta_0)}. \qquad (32)$$

From (7) and (32), the ASN function is

$$E_p(x, \theta; n) \qquad (33)$$
$$= \frac{L_p(\theta) \log \dfrac{\beta}{1 - \alpha} + [1 - L_p(\theta)] \log \dfrac{1 - \beta}{\alpha}}{p(x, \theta) \log \dfrac{p(x, \theta_1)}{p(x, \theta_0)} + [1 - p(x, \theta)] \log \dfrac{1 - p(x, \theta_1)}{1 - p(x, \theta_0)}}.$$

Both the numerator and denominator of (33) are functions of the slicing threshold x. At the parameters θ_0 and θ_1, recalling that $L_{p_0}(\theta_0) = 1 - \alpha$ and $L_{p_1}(\theta_1) = \beta$, (33) becomes

$$E_p(x, \theta_0; n) \qquad (34)$$
$$= \frac{(1 - \alpha) \log \dfrac{\beta}{1 - \alpha} + \alpha \log \dfrac{1 - \beta}{\alpha}}{p(x, \theta_0) \log \dfrac{p(x, \theta_1)}{p(x, \theta_0)} + [1 - p(x, \theta_0)] \log \dfrac{1 - p(x, \theta_1)}{1 - p(x, \theta_0)}}$$

and

$$E_p(x, \theta_1; n) \qquad (35)$$
$$= \frac{\beta \log \dfrac{\beta}{1 - \alpha} + (1 - \beta) \log \dfrac{1 - \beta}{\alpha}}{p(x, \theta_1) \log \dfrac{p(x, \theta_1)}{p(x, \theta_0)} + [1 - p(x, \theta_1)] \log \dfrac{1 - p(x, \theta_1)}{1 - p(x, \theta_0)}}.$$

At the indeterminate point,

$$p(x, \theta') = \frac{\log \dfrac{1 - p(x, \theta_0)}{1 - p(x, \theta_1)}}{\log \dfrac{p(x, \theta_1)}{p(x, \theta_0)} + \log \dfrac{1 - p(x, \theta_0)}{1 - p(x, \theta_1)}}, \qquad (36)$$

the ASN function can be obtained from (8). For this case it is easy to show that

$$E_p(x, \theta'; n) = \frac{\left(\log \dfrac{1 - \beta}{\alpha}\right)\left(\log \dfrac{1 - \alpha}{\beta}\right)}{\left(\log \dfrac{p(x, \theta_1)}{p(x, \theta_0)}\right)\left(\log \dfrac{1 - p(x, \theta_0)}{1 - p(x, \theta_1)}\right)}. \qquad (37)$$

Eq. (37) gives the maximum value of the ASN function. Applying the a, b, c transformations of (19)-(21) to (33), yields the equivalent expression in terms of the parameter u. Thus,

$$E_p(x, u; n) = \frac{bL_p(u) + a[1 - L_p(u)]}{(1 + c)p(x, u) - 1}. \qquad (38)$$

At the indeterminate point $u = 1$, (36) gives

$$p(x, 1) = \frac{1}{c + 1}; \qquad u = 1 \qquad (39)$$

and (37) becomes

$$E_p(x, 1; n) = -\frac{ab}{c}. \tag{40}$$

For a given value of the slicing threshold x, the ASN function is given as a function of u, where $0 \leq u \leq \infty$. From (30), the parameter p of the Bernoulli distribution is given as a function of u. It is now possible to obtain the ASN function in terms of the parameter p. For a given threshold x, the Bernoulli parameter p is related to the parameter θ of the probability density function $P(x, \theta)$ by (26). This gives the ASN function for the Bernoulli case as a function of θ.

Let us now refer to the ASN functions (34) and (35). It is seen that the numerator is only a function of the conditional error probabilities α, β. The denominator is a function of the parameters of the Bernoulli distribution. However, we repeat again, the parameters of the Bernoulli distribution $p(x, \theta_0)$, $p(x, \theta_1)$ are functions of the parameters θ_0 and θ_1 of the probability density of the random variable before slicing at the threshold x. The Bernoulli sequential test, corresponding to θ_0 and θ_1, is also a function of the slicing threshold x.

We have previously defined the denominator $E_\theta(z)$ of the ASN function as the average information per sample gained from measurement on condition that θ is the parameter of the probability density. This function is the average value of the measured quantity which is used for making a decision. It is seen that the larger the information gain per sample, the smaller the average sample number for a given set of parameters. Intuitively one would expect these quantities to be related in this manner. The function has important properties in its own right which make it useful as an information measure, exclusive of sequential analysis [6], [8], and [9]. For α, $\beta \ll 0.5$, the practical range of interest, it is seen from (34) and (35) that α, β fall off exponentially with increase in the average number of samples and with increase in the information gain. More precisely

$$\beta = \exp[E_p(x, \theta_0; n)E_p(x, \theta_0; z)]; E_p(x, \theta_0; z) < 0. \tag{41}$$

and

$$\alpha = \exp[-E_p(x, \theta_0; n)E_p(x, \theta_1; z)]; E_p(x, \theta_1; z) > 0. \tag{42}$$

(See [12] for the proof that $E_{\theta_0}(z) < 0$ and $E_{\theta_1}(z) > 0$.) Eq. (34) and (35), in a very compact and simple equation, relate the conditional error probabilities, the average number of samples required for termination at $p(x, \theta_0)$ and $p(x, \theta_1)$, and the average information per sample gained from measurement. It is also seen that the information gain function, (32), is a function of the slicing threshold x.

In order to compare the Bernoulli detector for a given slicing threshold x to the sequential test before slicing [corresponding to the density $P(x, \theta)$) at the same parameters $(\theta_0, \theta_1, \alpha, \beta)$], we take the ratio of the two ASN functions. That is, the efficiency $\delta_p(x, \theta_0)$ of the Bernoulli

test, on condition that θ_0 is the true parameter, is given by the ratio of (9) and (34), or

$$\delta_p(x, \theta_0) \tag{43}$$

$$= \frac{p(x, \theta_0) \log \dfrac{p(x, \theta_1)}{p(x, \theta_0)} + [1 - p(x, \theta_0)] \log \dfrac{1 - p(x, \theta_1)}{1 - p(x, \theta_0)}}{E_{\theta_0}(z)},$$

and, on condition that θ_1 is true, we have

$$\delta_p(x, \theta_1) \tag{44}$$

$$= \frac{p(x, \theta_1) \log \dfrac{p(x, \theta_1)}{p(x, \theta_0)} + [1 - p(x, \theta_1)] \log \dfrac{1 - p(x, \theta_1)}{1 - p(x, \theta_0)}}{E_{\theta_1}(z)}.$$

The efficiency of the Bernoulli detector is clearly a function of the slicing threshold x. For each Bernoulli test which corresponds to a different slicing threshold x, we obtain a different efficiency. It therefore seems reasonable to find the slicing threshold x which maximizes the efficiency $\delta_p(x, \theta_0)$ or $\delta_p(x, \theta_1)$. Since the denominators in (43) and (44) are independent of x, this is equivalent to maximizing the numerator or what has been defined in (32) as the information gain function. From (34) it is seen that maximizing the information gain with respect to x for given parameters $(\theta_0, \theta_1, \alpha, \beta)$ is equivalent to minimizing the ASN function on condition that θ_0 is true. Similarly, maximizing the information gain with respect to x for given parameters $(\theta_0, \theta_1, \alpha, \beta)$ minimizes the ASN function on condition that θ_1 is the true parameter. The Bernoulli detector chosen in this manner is optimum in the sense described.

If desired, the efficiency can also be defined at the point $\theta = \theta'$, where the ASN function is a maximum. From (8) and (27) we have

$$\delta(x, \theta') = \frac{\left(\log \dfrac{p(x, \theta_1)}{p(x, \theta_0)}\right)\left(\log \dfrac{1 - p(x, \theta_1)}{1 - p(x, \theta_0)}\right)}{E_{\theta'}(z^2)}. \tag{45}$$

More precisely, corresponding to the parameters $(\theta_0, \theta_1, \alpha, \beta)$, we choose the optimum slicing threshold at that value of x for which

$$\frac{d\delta(x, \theta)}{dx} = 0, \tag{46}$$

at one of the points $\theta = \theta_0$, $\theta = \theta_1$, or $\theta = \theta'$. When $(\theta_1 - \theta_0) \ll 1$, the threshold x is optimum in the interval $\theta_0 \leq \theta \leq \theta_1$. The optimization (46) for given $(\theta_0, \theta_1, \alpha, \beta)$ is equivalent to finding the threshold x which minimizes the ASN function, or

$$\frac{dE_p(x, \theta; n)}{dx} = 0 \tag{47}$$

at one of the points $\theta = \theta_0$, $\theta = \theta$, or $\theta = \theta'$.

E. Threshold Signal Approximation to Information Gain Function

The information gain function for a Bernoulli detector at the parameters $p(x, \theta_0)$ and $p(x, \theta_1)$ is given by the

enominator of (34) and (35), respectively. For the hreshold signal case, $(\theta_1 - \theta_0) \ll 1$, it is desirable to btain an approximation to the information gain function. n the problems considered here, we assumed that the one arameter family distributions are such that, when $\theta_1 - \theta_0) \ll 1$, $[p(x, \theta_1) - p(x, \theta_0)] \ll 1$ for all x. For onvenience, we will rewrite (32)

$$E_p(z) = p \log \frac{p_1}{p_0} + [1 - p] \log \frac{1 - p_1}{1 - p_0}, \qquad (48)$$

emembering that p, p_0, and p_1 are functions of θ, θ_0, and ₁ respectively and of x.

In (48) let us expand $\log p_1/p_0$ in a Taylor series,

$$\log \frac{p_1}{p_0} = \frac{\Delta p_0}{p_0} - \frac{1}{2} \frac{\Delta p_0^2}{p_0^2} + \cdots \qquad (49)$$

vhere

$$\Delta p_0 = p_1 - p_0.$$

By a similar expansion, $\log (1 - p_1/1 - p_0)$ is given by

$$\log \frac{1 - p_1}{1 - p_0} = -\frac{\Delta p_0}{1 - p_0} - \frac{1}{2} \frac{\Delta p_0^2}{(1 - p_0)^2} + \cdots \qquad (50)$$

f we now substitute these expansions into (48) and neglect third order terms in Δp_0, we obtain

$$E_p(z) \approx \frac{p_1 - p_0}{p_0(1 - p_0)} \left(p - \frac{p_1 + p_0}{2} \right). \qquad (51)$$

When $p = p_0$ and $p = p_1$, remembering that these depend on θ_0, θ_1, and x, we obtain from (51)

$$E_p(x, \theta_0; z) = -E_p(x, \theta_1; z)$$

$$= -\frac{1}{2} \frac{[p(x, \theta_1) - p(x, \theta_0)]^2}{p(x, \theta_0)[1 - p(x, \theta_0)]}. \qquad (52)$$

Hence, the information gain per sample at θ_0 and θ_1 is he same for threshold signals, for example, when $\theta_1 - \theta_0) \ll 1$.

The problem of detecting a sine-wave carrier of small ignal to noise ratio in Gaussian noise by Bernoulli trials vas treated by Harrington [7], who considers this problem or the case of fixed samples. For a given minimum detectable signal, he chooses the optimum slicing threshold : as the point at which the function

$$\rho(x) = \frac{p(x, \theta_1) - p(x, \theta_0)}{\sqrt{p(x, \theta_0)[1 - p(x, \theta_0)]}} \qquad (53)$$

s a maximum. For threshold signals, it is seen that the nformation gain criterion used here is related to that of 53) by

$$E_{p_0}(x, \theta_0; z) = -E_{p_1}(x, \theta_1; z) = \frac{\rho^2(x)}{2}. \qquad (54)$$

The maximum of $\rho(r)$ and $\rho^2(r)$ occurs at the same point, ince both functions are positive for all values of r. It is, herefore, seen that, for threshold signals, a choice of the ptimum slicing threshold based on a maximization of (x) is equivalent to a maximization of the information

gain function $E_{p_0}(z)$. Furthermore, the threshold is also optimum at the point p_1 and it can be shown that it is optimum for all $p_0 \le p \le p_1$. For large signal-to-noise ratios, the two criteria do not yield the same results. For certain distributions, it is possible for $\rho(r)$ to be infinite for certain sets of parameters, while the criteria presented here leads to finite results and has a maximum.
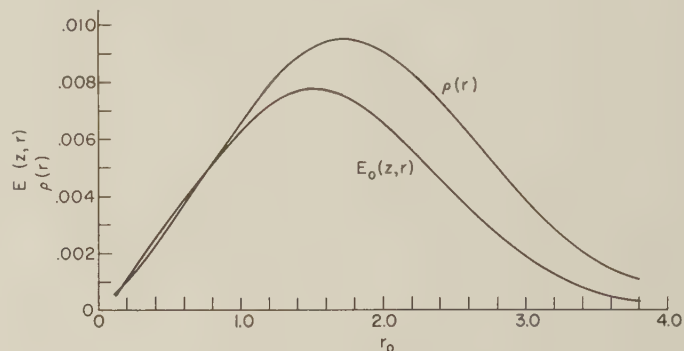

Fig. 5—Information gain function of Bernoulli sequential filter for a Gaussian signal in Gaussian noise of signal-to-noise ratio 0.50.

Fig. 5 is a curve of $E_{p_0}(r, z)$ and $\rho(r)$ for a Gaussian signal in Gaussian noise for a signal-to-noise ratio $\eta = 0.5$. $E_{p_0}(r, z)$ is a maximum at $r = 1.55$, while $\rho(r)$ is maximum at $r = 1.75$. Fig. 6 is a curve of the same function for
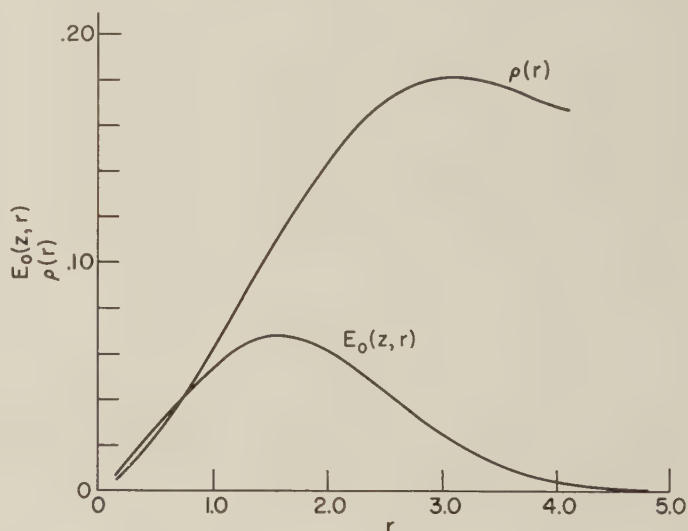

Fig. 6—Information gain function of Bernoulli sequential filter for a Gaussian signal in Gaussian noise of signal-to-noise ratio 1.0.

$\eta = 1$. The difference between the two maxima is quite large for this case. For the Gaussian case, it can be shown graphically that, for η somewhat greater than unity, $\rho(r)$ does not converge. The trend is clear. For the Rayleigh case, it can easily be shown that $\rho(r)$ diverges for all $\eta > \sqrt{2}$. It is felt that the comparison is a good illustration of the importance of the information gain criterion.

IV. Application to the Detection of a Sine-wave Carrier in Noise of Signal-To-Noise Ratio Less Than Unity

The important problem of designing a Bernoulli sequential filter for the detection of a sine-wave carrier

in noise is now considered. The Bernoulli detector tests the hypothesis $p = p_0$ that noise is present against the alternative hypothesis $p \geq p_1$ that signal-plus-noise is present. The distribution function of the envelope of a sine-wave carrier in noise is given in the literature [2]-[5] and [7].

$$p(r, \eta) = \int_r^\infty y e^{-\frac{1}{2}(y^2 + \eta^2)} I_0(\eta y) \, dy, \qquad (55)$$

$\eta =$ peak-signal to rms noise-ratio,

$I_0(\eta r) =$ zero order modified Bessel Function.

Corresponding to testing the hypothesis $\eta = 0$ against $\eta \geq \eta_1$, we define a Bernoulli test, $p(r, 0)$ against $p(r, \eta_1)$. For the case when only noise is present, $\eta = 0$ and

$$p(y, 0) = e^{-\frac{1}{2}r^2}. \qquad (56)$$

Also for small signal-to-noise ratios $\eta = \eta_1 < 1$, (55) can be approximated by expanding the integrand in a power series in η_1 and neglecting all orders of η_1 higher than the second. This yields

$$p(r, \eta_1) \approx \int_r^\infty y e^{-\frac{1}{2}y^2} (1 - \tfrac{1}{2}\eta_1^2 y^2 + \tfrac{1}{8}\eta_1^4 y^4 + \cdots + \cdots)$$

$$\left(1 + \frac{\eta_1^2 y^2}{4} + \frac{\eta_1^4 y^4}{64} + \cdots + \right) dy$$

$$\approx e^{-\frac{1}{2}r^2} \left(1 + \frac{r^2 \eta_1^2}{4}\right) = p(r, 0)\left[1 + \frac{r^2 \eta_1^2}{4}\right]. \qquad (57)$$

From (43) and (52) and using the results of Bussgang and Middleton [5] for $E_\eta(z)$, $\eta < 1$, the efficiency of the Bernoulli detector, as compared to the best detector given by the theory for threshold signals, is found to be

$$\delta_p(r, \eta_1) = \delta_p(r, 0) = \frac{4[p(r, \eta_1) - p(r, 0)]^2}{p(r, 0)[1 - p(r, 0)]\eta_1^4}. \qquad (58)$$

$\delta_p(r, \eta_1) = \delta_p(r, 0) =$ efficiency of Bernoulli detector at the output of a slicing operator at the threshold $x = r$, when $\eta = \eta_1$ and $\eta = 0$ are the threshold parameters. Substituting (57) into (58) yields for the efficiency

$$\delta_p(r, \eta_1) = \delta_p(r, 0) = \frac{p(r, 0)}{1 - p(r, 0)} \frac{r^4}{4}. \qquad (59)$$

We can eliminate r in (59) by making use of (56). Hence,

$$\delta_p(r, \eta_1) = \delta_p(r, 0) = \frac{p(r, 0)[\log p(r, 0)]^2}{1 - p(r, 0)}. \qquad (60)$$

Since $p(r, 0)$ is a monotonic function of r, maximizing (60) with respect to p is the same as maximizing with respect to r. The value of $p(r, 0)$ which maximizes (60) is given by

$$p(r, 0) = e^{-2[1 - p(r, 0)]} \qquad (61)$$

or

$$p(r, 0) = 0.205.$$

The corresponding threshold is obtained from (56) as

$$r = \frac{R}{N} = 1.78. \qquad (62)$$

$R =$ voltage sample of carrier envelope.

$N =$ rms value of noise.

The efficiency at the optimum threshold is given by

$$\delta_p(\eta_1) = \delta_p(0) = 64 \text{ per cent.} \qquad (63)$$
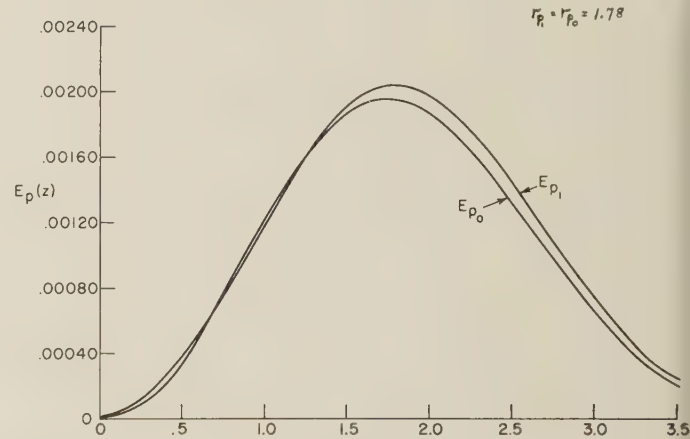

Fig. 7—Information gain function of Bernoulli sequential filter for a sine-wave carrier in noise of signal-to-noise ratio 0.50.
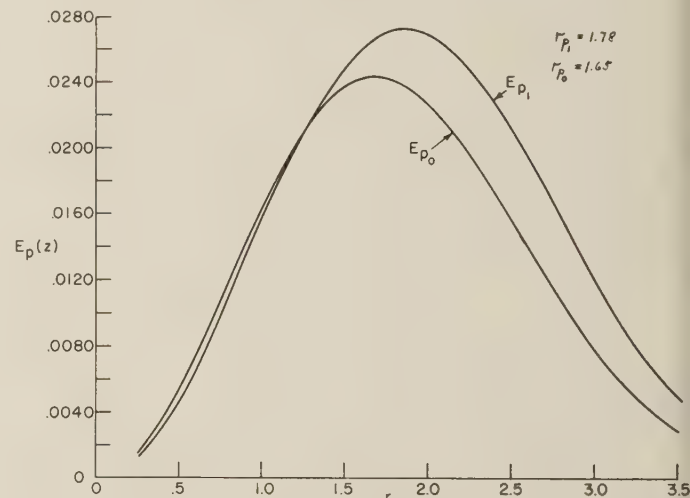

Fig. 8—Information gain function of Bernoulli sequential filter for a sine-wave carrier in noise of signal-to-noise ratio 1.0.
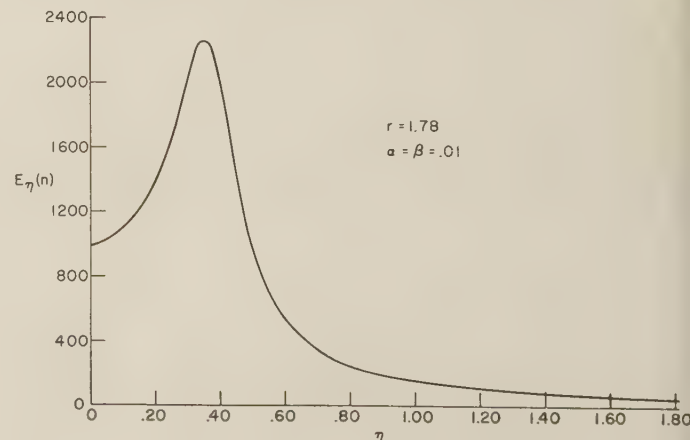

Fig. 9—Average sample number function of Bernoulli sequential filter for a sine-wave carrier in noise of signal-to-noise ratio 0.50.
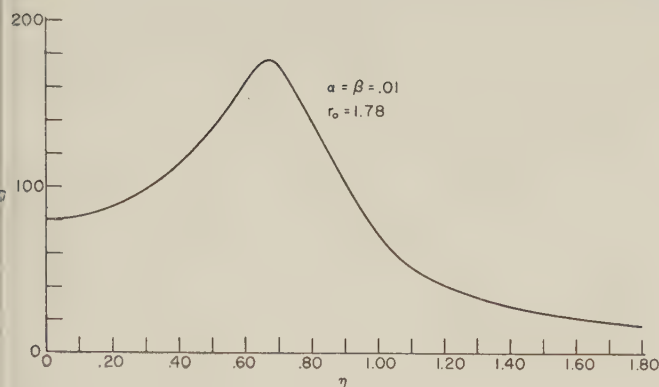
ig. 10—Average sample number function of Bernoulli sequential filter for a sine-wave carrier in noise of signal-to-noise ratio 1.0.
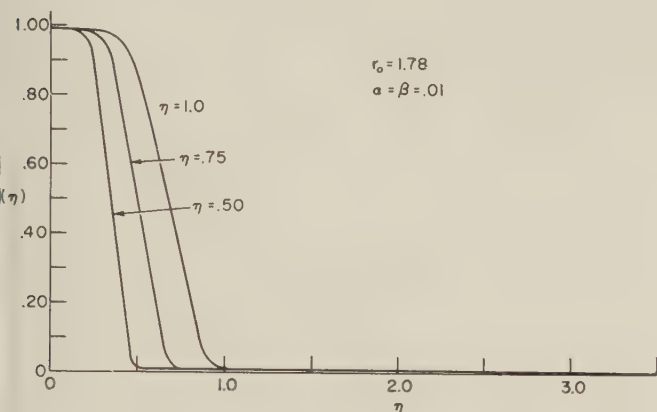

ag. 11—Operating characteristic function of Bernoulli sequential filter for a sine-wave carrier in noise.

his checks the results given elsewhere [5] for the same roblem.

Figs. 7 and 8, p. 130, are curves of the information gain unction at the threshold parameters for signal-to-noise

ratios $\eta_1 = 0.5$; and 1. Fig. 9, p. 130, and Fig. 10 are curves of the ASN function for $\eta_1 = 0.5$, 1 and $\alpha = \beta = 0.01$. Fig. 11 gives curves of the OC function for $\eta_1 = 0.5$, 0.75, 1 and $\alpha = \beta = 0.01$.

Acknowledgment

Bibliography

[1] Blackwell, and Girshick, M. A. *Theory of Games and Statistical Decisions*, New York: John Wiley and Sons, Inc., 1954.
[2] Blasbalg, H. *Bernoulli Sequential Filtering*, TR No. AF-27, Baltimore: The Johns Hopkins University, Radiation Laboratory, January, 1956.
[3] Blasbalg, H. *Theory of Sequential Filtering and Its Application to Signal Detection and Classification*, TR No. AF-8, Baltimore: The Johns Hopkins University, Radiation Laboratory, October 18, 1954.
[4] Blasbalg, H. "Theory of Sequential Filtering and Its Application to Signal Detection." Doctor of Engineering Thesis, Department of Engineering, The Johns Hopkins University.
[5] Bussgang, J. J. and Middleton, D. "Optimum Sequential Detection of Signals in Noise," IRE Transactions on Information Theory, Vol. IT-1 (December, 1955), pp. 5-18.
[6] Chernoff, H. "Large Sample Theory; Parameter Case," *The Annals of Mathematical Statistics*, Vol. 27, (March, 1956), pp. 1-20.
[7] Harrington, J. V. "An Analysis of the Detection of Repeated Signals in Noise by Binary Integration," IRE Transactions on Information Theory, Vol. IT-1 (March, 1955), pp. 1-9.
[8] Kullback, S. and Leibler, R. A. "Information and Sufficiency," *The Annals of Mathematical Statistics*, Vol. 22, (1951) pp. 79-86.
[9] Kullback, S. "Certain Inequalities in Information Theory and the Cramer-Rao Inequality," *The Annals of Mathematical Statistics*, Vol. 25 (December, 1954), pp. 745-752.
[10] Reed, I. S. and Dinneen, G. P. "An Analysis of Signal Detection and Location by Digital Methods," IRE Transactions on Information Theory, Vol. IT-2 (March, 1956), pp. 29-38.
[11] Shwartz, M. "A Coincidence Procedure for Signal Detection," IRE Transactions on Information Theory, Vol. IT-2 (December, 1956), pp. 135-139.
[12] Wald, A. *Sequential Analysis*, New York: John Wiley and Sons, Inc., 1947.

A Theory of Weighted Smoothing*

LOUIS A. ULE†

Summary—The problem attacked in this paper is that of a system ith a stationary random error input, a nonrandom signal input and nonrandom error input. The output of the system is required to ave a minimum weighted response to the random error and otherise to consist only of arbitrary functions of time linearly related to he nonrandom signal input. The more general case, which includes random signal as well, is not considered.

Introduction

IT HAS LONG been known that it is possible to give different values of weight, depending on the source, time, medium, and other circumstances, to pieces f evidence from which a particular conclusion is to be rawn, independently of the reasoning process used to

arrive at the conclusion. In data smoothing the evidence to be so weighted is the input signal to the system, and the conclusion to be drawn, or output signal, is some function of the smoothed data. In the weighted smoothing considered in this paper, the input data will be weighted temporally only. Typically, though by no means necessarily, recent data will be given more weight than old data.

The concept of weighted smoothing is not new.[1] Filters, for example, which consider inputs occurring within only the past T seconds,[2] may be said to give such data an

[1] L. A. Ule, "Weighted least-squares smoothing filters," IRE Trans., vol. CT-2, pp. 197–203; June, 1955.
[2] L. A. Zadeh and J. R. Ragazzini, "An extension of Wiener's theory of prediction," *J. Appl. Phys.*, vol. 21, pp. 645-655, July, 1950.

even weight and to any older inputs zero weight. This data weighting is not to be confused with the weighting function of a linear system; the latter determines the type of conclusion to be drawn from the input although its exact form will depend on the way the input is to be weighted.

Since the various theories regarding smoothing are optimum solutions, they can differ only in the nature of the problem attacked. These problems differ in the nature of what is known *a priori* about the input, the way this input is to be weighted temporally and the type of output that is desired. The Wiener[3] filter, for example, assumes that the input, both signal and error, is a stationary time series. It assigns all past inputs uniform weight and future inputs zero weight. The output is required to differ from the signal part of the input with a least average squared error. The filters of Zadeh and Ragazzini[4] extend the input to include power series in time of finite degree n, give inputs up to T seconds old even weight and all other inputs zero weight. The output is required to differ least from any linear operation on the input.

Formulation

In this paper, the input will be weighted nonuniformly, and within limits, in any desired manner. This weighting will be called the memory of the filter to distinguish it from the system weighting function which performs other operations in addition to weighting. The input will be assumed to consist of the sum of three kinds of components:

1) A linear combination of r known functions of time, described either by mathematical equations or by graphs, and corresponding to the nonrandom components of the signal.

2) A linear combination of $n - r$ known functions of time, which with the r functions above are linearly independent within the memory of the filter. These functions correspond to the nonrandom part of the error input.

3) A stationary known error spectrum corresponding to the random part of the error input. A stationary random signal spectrum is not assumed since this case is covered by the Wiener filter with the addition of suitable constraints.

The output of the filter will consist of some response to the random error input and any r functions of time either separately or in summation proportional in amplitude to the r signal components of the input. These functions, r in number, may be the same as the signal input, they may be predictions of the signal input components not necessarily all by the same amount, they may be linear operations on these inputs not necessarily all the same or they may bear no functional relation to the input components other than that their amplitudes are each

proportional to one of the components of the input. The special case of zero output in the presence of a particular input is used to reject the undesired $n - r$ input components.

The desired signal inputs we shall call $f_i(t)$, $i = 1, 2, \cdots r$. The undesired components will be similarly designated but with $i = r + 1, r + 2, \cdots n$.

The random error input will be represented by its autocorrelation function $R(\tau)$. This function, by the Wiener-Khintchine relationship is the Fourier cosine transform of the power spectrum.

The desired output will be represented by the sum of functions $p_i(t)$, $i = 1, 2, \cdots n$, where $p_i(t) = 0$ for $i = r + 1, r + 2, \cdots n$. Some of the forms that these functions can assume are:

Simple estimation: $p_i(t) = f_i(t)$
Prediction: $p_i(t) = f_i(t + \alpha)$
Differentiation: $p_i(t) = df_i(t)/dt$
Attenuation or gain: $p_i(t) = kf_i(t)$
Arbitrary: For an input $a_i f_i(t)$ the output is $a_i p_i(t)$.

The amplitude of each output component is proportional to the amplitude of the associated input component. As will be seen later, these components can be obtained separately or in summation.

In producing the above outputs, there is no question of error, since we will require that they be produced exactly in the presence of the associated inputs and in the absence of noise. The only error that remains is the response to the random noise input. Even this latter can be made as small as desired by the error weighting function to be defined, but the resulting filter will have a correspondingly narrow bandwidth and long settling time. If a signal spectrum were involved, as in the Wiener filter, this would not be the case, since a filter with too narrow a bandwidth would produce increased error due to the attenuation of the random part of the signal.

As indicated, the error output of the filter is to be weighted temporally by a memory function prior to minimization. We represent this function by $H(\tau)$, where τ, measured from the present, represents time past. For example, τ would be 10 for a time 10 seconds ago, zero for the present, and minus 10 for a time 10 seconds from now. Intuitively, we will require $H(\tau)$ to be large for those times relative to the present when the signal input is the most significant, small when the signal input is likely to be obsolete or erroneous and zero when the signal input is known to be irrelevant. For a realizable filter $H(\tau)$ is zero for all future inputs: *i.e.*, $H(\tau) = 0$, for $\tau < 0$. The case where the memory function is time-varying will not be discussed here since it is a rather uncomplicated generalization.

The filter weighting function $W(\tau, t)$, as distinct from the memory function, will be a function of time t, since the generality of the signal inputs and outputs assumed here will in some cases require time-varying filters. For such a filter, the output $g(t)$ for an input $f(t)$ is obtained

[3] N. Wiener, "The Extrapolation, Interpolation, and Smoothing of Stationary Time Series," John Wiley and Sons, Inc., New York, N. Y.; 1949.
[4] Zadeh and Ragazzini, *loc. cit.*

om the filter weighting function by means of the con-
olution integral; *i.e.*,

$$g(t) = \int_0^\infty W(\tau, t) f(t - \tau) \, d\tau. \tag{1}$$

Since the filter will be constrained to produce its outputs
ithout error in the presence only of its proper inputs
(t), $i = 1, 2, \cdots n$, the only error that will otherwise
esult will be due to the random noise input. The ensemble
verage square of this error $\sigma^2(t)$ is as follows:

$$\sigma^2(t) = \int_0^\infty \int_0^\infty W(x, t) W(y, t) R(x - y) \, dx \, dy \tag{2}$$

here $R(\tau)$ is the autocorrelation function describing the
andom noise input. It is possible to make this error
rbitrarily small, yet meet all other requirements, by a
lter with a very long memory, or, what amounts to the
ame thing, by a filter with a very narrow bandwidth. A
ethod often used to increase the bandwidth and reduce
he settling time of the filter is to constrain the weighting
unction to be zero for $\tau > T$, where T is a finite number.
uch a filter will have a settling time no greater than T
econds but will require delay lines in its construction.
he method proposed here to obviate this difficulty is to
eight the error by the memory function $H(\tau)$ to produce
weighted error $\sigma_0^2(t)$ as follows:

$$\sigma_0^2(t) = \int_0^\infty \int_0^\infty \frac{W(x, t) W(y, t)}{H(x) H(y)} R(x - y) \, dx \, dy. \tag{3}$$

Since a subsequent minimization is implied, it is clear
hat if the weighted error is to be finite, let alone a mini-
um, $W(\tau, t)$ must be zero when $H(\tau)$ is zero. This
ntuitive requirement is the reason for putting the memory
unction into the denominator. By setting

$$H(\tau) = U(\tau) - U(\tau - T), \tag{4}$$

here $U(\tau)$ is the unit step function, we will, for example,
imit the filters memory to a duration of T seconds.

SOLUTION

Before proceeding to minimize the weighted error by
he calculus of variations we meet the other requirements
reviously set forth by imposing the following set of
onstraints.

$$\int_0^\infty W(\tau, t) f_i(t - \tau) \, d\tau = p_i(t) \qquad i = 1, 2, \cdots, n. \tag{5}$$

An infinite number of filters meet these constraints,
nd of this number we select the one for which the weighted
rror (3) is least. Applying the calculus of variations we
btain the following integral equation

$$\int_0^\infty \frac{W(x, t)}{H(x)} R(\tau - x) \, dx = \sum_{i=1}^n H(\tau) \lambda_i(t) f_i(t - \tau), \tag{6}$$

which together with constraints (5) determines the
eighting function $W(\tau, t)$ of the optimum filter. The
unctions $\lambda_i(t)$ are the n Lagrangian multipliers which
re eliminated from the $n + 1$ relations (5) and (6).

SOLUTION FOR WHITE NOISE

An explicit solution for white noise is obtained quite
readily since in this case $R(\tau) = \delta(\tau)$, where $\delta(\tau)$ is the
unit impulse. In this case (6) reduces to

$$W(\tau, t) = H^2(\tau) \sum_{i=1}^n \lambda_i(t) f_i(t - \tau), \tag{7}$$

substitution of which into (5) results in

$$\sum_{i=1}^n \lambda_i(t) \int_0^\infty H^2(\tau) f_i(t - \tau) f_j(t - \tau) \, d\tau = p_j(t), \tag{8}$$
$$j = 1, 2, \cdots, n,$$

a set of simultaneous equations in the unknown $\lambda_i(t)$.
By defining

$$\sum_{j=1}^n c_{jk}(t) \int_0^\infty H^2(\tau) f_i(t - \tau) f_j(t - \tau) \, d\tau = \delta_{ik}, \tag{9}$$

where δ_{ik} is the Kronecker delta, the numbers $C_{jk}(t)$
become the elements of the inverse matrix corresponding
to the solution, so that

$$\lambda_i(t) = \sum_{j=1}^n c_{ij}(t) p_j(t), \qquad i = 1, 2, \cdots, n. \tag{10}$$

Examination of (9) and (10) shows that the only
conditions that need be imposed on $H(\tau)$ is that the
matrix defined by the integral in (9) be nonsingular so
that a solution for the $\lambda_i(t)$ exists. Substitution of (10)
into (7) produces an explicit expression for the filter
weighting function; *i.e.*,

$$W(\tau, t) = H^2(\tau) \sum_{i=1}^n \sum_{j=1}^n C_{ij}(t) f_i(t - \tau) p_j(t). \tag{11}$$

This filter is time-varying. An exception occurs if the n
functions $f_i(t)$ are n linearly independent solutions to an
nth order linear homogeneous differential equation with
constant coefficients and the functions $p_i(t)$ can be
obtained by time-invariant operations on the n functions
$f_i(t)$.

SOLUTION FOR ARBITRARY NOISE

The simplicity of the solution for white noise suggests
the following approach in other cases. The filter is divided
into two cascaded networks, the first of which is devised
to turn the noise spectrum at its input into white noise
at its output. The second network then has a white noise
input. Constraints (5) are imposed upon the combined
network and the second filter is chosen to make the
weighted error due to the random noise input a minimum
subject to these constraints. The mean square response
to the random noise for the combined network is now as
follows

$$\sigma^2(t) = \int_0^\infty \cdots \int_0^\infty W_1(x) W_1(y) W_2(r, t) W_2(s, t)$$
$$\cdot R(x - y + r - s) \, dx \, dy \, dr \, ds, \tag{12}$$

where $W_1(\tau)$ is the known weighting function of the first
network and $W_2(\tau, t)$ is the yet to be determined weighting

function of the second network. As before, we will minimize the weighted error $\sigma_0^2(t)$, defined as follows,

$$\sigma_0^2(t) = \int_0^\infty \cdots \int_0^\infty W_1(x) W_1(y) \frac{W_2(r,\,t) W_2(s,\,t)}{H(r) H(s)}$$

$$\cdot R(x - y + r - s)\ dx\ dy\ dr\ ds, \qquad (13)$$

except that now the constraints become

$$\int_0^\infty \int_0^\infty W_1(x) W_2(y,\,t) f_i(t - x - y)\ dx\ dy = p_i(t),$$

$$i = 1, 2, \cdots, n. \qquad (14)$$

Applying the calculus of variations again and noting that the second network has a white noise input, the weighting function of this network becomes

$$W_2(\tau,\,t)$$

$$= H^2(\tau) \sum_{i=1}^n \lambda_i(t) \int_0^\infty W_1(x) f_i(t - \tau - x)\ dx. \qquad (15)$$

Eliminating the Lagrangian multipliers, as before, we obtain

$$W_2(\tau,\,t) = H^2(\tau) \sum_{i=1}^n \sum_{j=1}^n C_{ij}(t) p_j(t)$$

$$\cdot \int_0^\infty W_1(x) f_i(t - \tau - x)\ dx, \qquad (16)$$

where the numbers $c_{ij}(t)$ are the elements of the inverse matrix of the set of equations whose matrix elements are

$$b_{ij}(t) = \int_0^\infty \int_0^\infty H^2(z) W_1(x)$$

$$\cdot W_1(y) f_i(t - x - z) f_j(t - y - z)\ dx\ dy\ dz. \qquad (17)$$

As before, the combined filter is, in general, time-varying and of considerable complexity. It is, however, readily synthesized. Examination of the convolution integral (5) shows that factors of the weighting function which are functions of $t - \tau$ are synchronized with the input signal. The input signal is, therefore, simply multiplied by these factors before being acted upon by the convolution integral. Factors which are functions of τ only are time invariant and generally denote filter networks which can be synthesized through the Laplace transform. Finally, the remaining factors which are functions of t only can be factored out of the integral; they, therefore, multiply the output of the preceding filter sections.

The above is not the only mechanization possible. Functions of $t - \tau$ can often be factored into functions of t and τ only, or otherwise manipulated, so that when the preceding procedure is applied a different configuration results.

A second order mechanization is blocked out in Fig. 1. It is interesting to note that, other than in input network $W_1(\tau)$, the only smoothing network that appears consists of n identical filters whose weighting function is $H^2(\tau)$, the filter memory function squared. After passing through
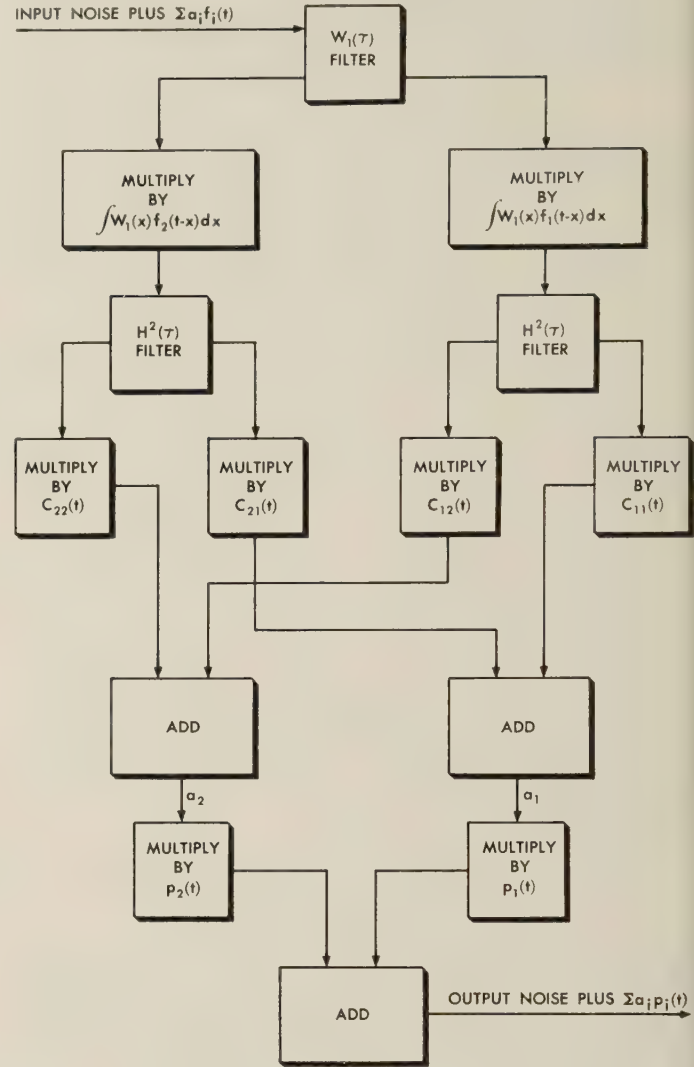
Fig. 1—Second order mechanization.

the input network $W_1(\tau)$, the input signal is multiplied by n multipliers, each multiplying by one of the functions $f_i(t)$ as modified by the input network. This matrix of n multipliers is then properly called an autocorrelation matrix. The n-multiplied signals are individually smoothed and then transformed to n constant voltages by the matrix of multipliers $C_{ij}(t)$. These n constant voltages are the coefficients a_i of the input $f(t)$; *viz.*,

$$f(t) = \sum_{i=1}^n a_i f_i(t). \qquad (18)$$

These dc voltages can be modulated into any output functions $p_i(t)$ desired by output multipliers and these outputs can be obtained separately, in summation, or only in part. As pointed out previously, there are occasions when no time-varying multiplications are required.

EXAMPLE

The example to be given will be that of a phase detector which will have a dc output equal to the peak value of the sine component of a random phase sine wave signal of one radian frequency, and no output to the cosine com-

onent. Of course, if desired, a dc output equal to the peak value of the cosine component can be obtained with no additional mathematical labor since the matrix inversion to be performed is the same regardless of the desired output. Specifications for this detector are as follows:

Nonrandom input: $a_1 \sin t + a_2 \cos t$

Noise spectral density:

$$\frac{\omega^2 + 1}{\omega^2 + 4}$$

Desired output: $p_1(t) = a_1$
$\qquad\qquad\quad p_2(t) = 0$

Filter memory function: $H(\tau) = e^{-0.05\tau}$.

The weighting function $W_1(\tau)$ which will make the noise input into the second network white is quite obviously the inverse Laplace transform form of

$$\frac{s + 2}{s + 1}$$

so that

$$W_1(\tau) = \delta(\tau) + e^{-\tau}.$$

Using (17) we next calculate the functions $b_{ij}(t)$. These are:

Fig. 2—Phase detector.

$$b_{11}(t) = 12.5 + \frac{1}{4.01}\,(1.4 \cos 2t - 2.075 \sin 2t)$$

$$b_{12}(t) = b_{21}(t) = -\frac{1}{4.01}\,(1.4 \sin 2t + 2.075 \cos 2t)$$

$$b_{22}(t) = 12.5 - \frac{1}{4.01}\,(1.4 \cos 2t - 2.075 \sin 2t).$$

The elements of the inverse matrix without approximation are:

$$c_{11}(t) = 0.0802 - 0.0016\,(1.4 \cos 2t - 2.075 \sin 2t)$$

$$c_{12}(t) = c_{21}(t) = 0.0016\,(1.4 \sin 2t + 2.075 \cos 2t)$$

$$c_{22}(t) = 0.0802 + 0.0016\,(1.4 \cos 2t - 2.075 \sin 2t).$$

The integrals of (16) are next evaluated for sint and cost. These are, respectively:

$$1.5 \sin (t - \tau) - 0.5 \cos (t - \tau)$$

and

$$1.5 \cos (t - \tau) + 0.5 \sin (t - \tau).$$

Substitution of the foregoing expressions into (16) results in

$$W_2(\tau, t)$$
$$= e^{-0.1\tau}\{[0.0802 - 0.0016\,(1.4 \cos 2t - 2.075 \sin 2t)]$$
$$\times [1.5 \sin (t - \tau) - 0.5 \cos (t - \tau)]$$
$$+ 0.0016\,[1.4 \sin 2t + 2.075 \cos 2t]$$
$$\times [1.5 \cos (t - \tau) + 0.5 \sin (t - \tau)]\}.$$

The complete phase detector is shown mechanized in Fig. 2. The mechanization shows two dc outputs equal to the peak of the sine and cosine components of the input sine wave. To meet the requirement of just one output, two multipliers and an adder are eliminated from the circuit.

Conclusion

A contribution of this paper lies in the extension of the class of nonrandom functions which can be used to represent a signal. These functions may be defined by a mathematical expression or by graphs and need only be finite and piecewise continuous. A practical aspect is the possible specification of the memory function $H(\tau)$ so as to obtain lumped constant filters as against the conventional specification that $H(\tau) = U(\tau) - U(\tau - T)$, which requires a filter with distributed constants. Finally, a way has been devised to apply the weighted error concept to filters having an arbitrary stationary noise input. The generalization could have been carried further to include nonstationary time series and a time-varying memory function. At this stage of the art it is sufficient to propose the problem; its solution is a foregone conclusion.
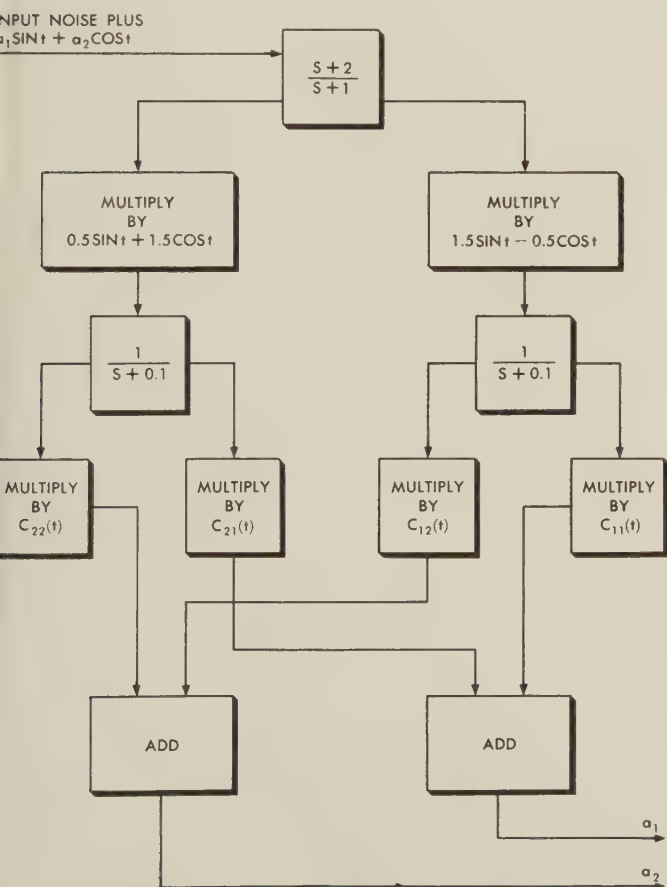
The Response of a Phase-Locked Loop to a Sinusoid Plus Noise*

STEPHEN G. MARGOLIS†

Summary—The phase-locked loop is a practical device for separating a sinusoidal signal from additive noise. In this device the incoming signal-plus-noise is multiplied by a noise-free sinusoid generated by a voltage-controlled oscillator (vco). The filtered product is used to lock the phase of the vco output to that of the incoming signal, thus producing a relatively clean version of the incoming signal in which the noise manifests itself as a small phase modulation. Analysis of this noise-produced phase modulation is complicated by the presence of the multiplier at the input to the loop. This paper presents a perturbation method which reduces this inherently nonlinear servo analysis problem to the analysis of a series of linear systems, the first of which is related to the linear model used by previous authors. The perturbation technique permits the phase modulation resulting from an arbitrary noise input to be computed to any desired accuracy. This analysis is particularly useful in predicting loop performance when it is used as a narrow-band receiver in a phase-comparison angle-measuring system.

LIST OF SYMBOLS

THE FOLLOWING is a list of the symbols used in this paper:

A = peak amplitude of sinusoidal input, volts.

A_1 = peak amplitude of sinusoidal input to reference channel, volts.

$A_m(\theta_s)$ = peak amplitude of sinusoidal input to measurement channel, volts.

C = capacitance used in compensating network, farads.

$f(t)$ = random time-function, dimensionless.

$g(t)$ = random time function defined in (30c) dimensionless.

$H(S)$ = transfer function of phase-locked loop, relating small transient changes in input phase angle to the resulting changes in output phase angle.

$h(\tau)$ = inverse L-transform of $H(S)$.

$J = \dfrac{2N}{A}$ = noise-to-signal ratio, dimensionless.

$K = \dfrac{GA}{2} \times \dfrac{(1 \text{ radian})}{\text{volt-second}}$ = loop gain constant, radians/second.

N = noise amplitude factor, volts.

N_1 = noise amplitude factor for reference channel, volts.

$N_m(\theta_N)$ = noise amplitude factor for measurement channel, volts.

R_1, R_2 = resistances used in compensating network, ohms.

S = complex frequency, radians/second.

t = time, seconds.

T = averaging interval, seconds.

$T_1 = (R_1 + R_2)C$ = larger time-constant of compensating network seconds.

$T_2 = R_2C$ = smaller time-constant of conpensating network, seconds.

$v_0(t)$ = output of voltage-controlled oscillator, volts.

$v_1(t)$ = input to phase-locked loop, volts.

$v_2(t)$ = output of multiplier, volts.

$v_4(t)$ = input to control terminal of voltage-controlled oscillator, volts.

$v_5(t)$ = output of angle-measuring system, volts.

$v_m(t)$ = input to measurement channel from antenna, volts.

$v_r(t)$ = input to measurement channel from reference channel, volts.

W = noise spectral density, $\dfrac{(\text{volts})^2\text{-second}}{\text{radian}}$.

$\Delta\omega$ = half-bandwidth of narrow-band noise, radians/second.

$\Delta\omega' = \dfrac{\Delta\omega}{K}$ = normalized half-bandwidth of narrow-band noise, dimensionless.

θ_N = angle between reference line and line joining noise-source location and antenna location, radians.

θ_s = angle between reference line and line joining signal-source location and antenna location, radians.

$\lambda(t) = \lambda_{vco} - \lambda_i$ = noise-produced phase error, radians.

$\lambda_1(t), \lambda_2(t) \cdots \lambda_n(t)$ = components of $\lambda(t)$, radians.

$\lambda_i(t)$ = phase angle of input sinusoid, radians.

$\Lambda_i(s)$ = L-transform of $\lambda_i(t)$.

$\lambda_{vco}(t)$ = phase angle of voltage-controlled oscillator output, radians.

$\Lambda_{vco}(s)$ = L-transform of $\lambda_{vco}(t)$.

σ_1 = rms noise level at input to reference channel, volts.

σ_f = rms value of $f(t)$, dimensionless.

σ_r = rms noise level at input to measurement channel, volts.

τ = dummy variable used in convolution integral, seconds.

$\omega_0 = \omega_c - \omega_1$ = difference between center-frequency of noise band and angular frequency of input sinusoid, radians/second.

$\omega_0' = \dfrac{\omega_0}{K}$ = normalized difference frequency, dimensionless.

ω_1 = angular frequency of input sinusoid; center-frequency of voltage controlled oscillator, radians/second.

ω_c = center-frequency of narrow noise band, radians/second.

ω_N = closed-loop bandwidth of compensated phase-locked loop, radians/second.

INTRODUCTION

The basic function of a phase-locked loop is to lock the phase of the output of a voltage-controlled oscillator (vco)[1] to the phase of an incoming sinusoidal signal. A properly designed loop is capable of performing this function even in the presence of noise power which greatly exceeds the signal power.[2] The noise manifests itself as a random phase modulation of the vco output. This random phase modulation is usually treated by deriving an equivalent transfer function for the loop, by which transient changes in input phase can be related to the resulting changes in output phase; the incoming signal-plus-noise is usually approximated by a signal phase-modulated by noise, and the output phase modulation is then found by passing the input phase modulation through the equivalent transfer function. This paper takes a somewhat more direct approach to the problem, and thus avoids the necessity of converting a signal-plus-noise to a signal phase-modulated by noise; consequently avoiding defining equivalent phase angle when the noise power greatly exceeds the signal power. The results are easily interpreted in terms of the noise-free behavior of the loop, which will be reviewed first.
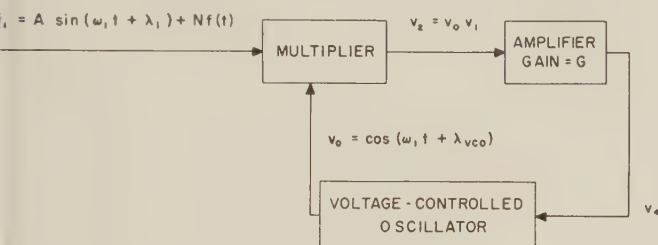

Fig. 1—Basic structure of a phase-locked loop.

Fig. 1 shows the basic structure of a phase-locked loop with a noisy input. Assuming for the moment that $N = 0$ (*i.e.*, the input signal is free from noise) and neglecting terms with frequency $2\omega_1$, the output of the amplifier is

$$v_2 = \frac{GA}{2} \sin (\lambda_i - \lambda_{\text{vco}}) \doteq \frac{GA}{2} (\lambda_i - \lambda_{\text{vco}})$$

$$\text{for} \quad (\lambda_i - \lambda_{\text{vco}}) \ll \frac{\pi}{2}. \quad (1)$$

It is assumed that the sensitivity of the vco is 1 radian per second per volt,[3] so that

$$\dot{\lambda}_{\text{vco}} = v_4 \times \frac{1 \text{ radian}}{\text{volt-second}}. \quad (2)$$

Combining (1) and (2), the equation relating the vco phase to the input phase is[4]

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[1] In this paper, the term voltage-controlled oscillator refers to an oscillator in which the deviation of the output *frequency* from its nominal value is proportional to a control voltage.

[2] R. Jaffee and E. Rechtin, "Design and performance of phase-locked circuits capable of near optimum performance over a wide range of input signal and noise levels," IRE TRANS., Vol. IT-1, p. 66; March, 1955.

[3] The actual numerical value of the vco sensitivity can be absorbed into the gain G.

[4] By definition, K has the units radians/second.

$$\dot{\lambda}_{\text{vco}} + K\lambda_{\text{vco}} = K\lambda_i \quad (3)$$

where

$$K = \tfrac{1}{2}GA \times \frac{1 \text{ radian}}{\text{volt-second}}.$$

The steady-state solution is $\lambda_{\text{vco}} = \lambda_i$. The response of this simple loop to small changes in the input phase λ_i can be found by L-transforming (3). The result is

$$H(S) = \frac{\Lambda_{\text{vco}}(S)}{\Lambda_i(S)} = \frac{K}{S + K} \quad (4)$$

which defines the equivalent transfer function for the loop. Here Λ_{vco} and Λ_i are the transforms of λ_{vco} and λ_i, respectively.

In the simple loop the constant K determines both the bandwidth of the transfer function and the range of frequencies over which the loop will lock (pull-in range).[5] These parameters may be controlled separately by including an RC compensating network, connected between
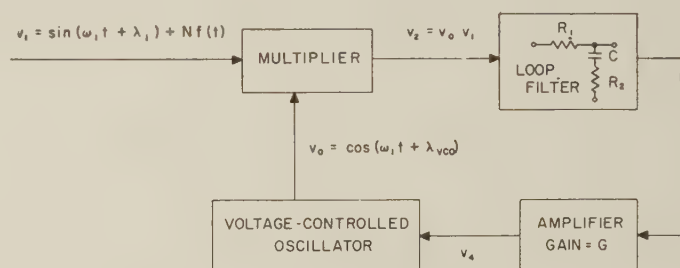
Fig. 2—Phase-locked loop with RC compensation.

the multiplier output and the vco input, as shown in Fig. 2. For the compensated loop.[5,6]

$$H(S) = \frac{\Lambda_{\text{vco}}(S)}{\lambda_i(S)} = \frac{\dfrac{KT_2}{T_1} S + \dfrac{K}{T_1}}{S^2 + \left(\dfrac{1}{T_1} + \dfrac{KT_2}{T_1}\right)S + \dfrac{K}{T_1}} \quad (5)$$

where

$$T_1 = (R_1 + R_2)C$$

$$T_2 = R_2C$$

and

$$K = \tfrac{1}{2}GA \times \frac{1 \text{ radian}}{\text{volt-second}} ,$$

as before. Here again, the steady-state solution is $\lambda_{\text{vco}} = \lambda_i$ and the pull-in range is $\pm K$ radians per second, but the bandwidth of the transfer function ω_N is equal to $\sqrt{K/T_1}$. For properly damped response to changes in λ_i it is usual to choose

$$\frac{KT_2}{T_1} \doteq 1.5\omega_N$$

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[5] W. J. Gruen, "Theory of afc synchronization," PROC. IRE, vol. 41, pp. 1043-1048; September, 1953. See pp. 1045-1046.

[6] P. F. Ordung, J. E. Gibson, and B. J. Shinn, "Closed Loop Automatic Phase Control," presented at AIEE Summer and Pacific General Meeting, Los Angeles, Calif.; June 21-25, 1954.

and to make

$$\frac{KT_2}{T_1} \gg \frac{1}{T_1}.$$

THE ANALYSIS OF NOISE IN SIMPLE LOOPS

Referring again to Fig. 1, the input to the simple loop is

$$v_1 = A \sin(\omega_1 t + \lambda_i) + Nf(t). \qquad (6)$$

For convenience in the subsequent analysis, the noise has been written as the product of a constant N and a time-varying part $f(t)$. The vco output is

$$v_0 = \cos(\omega_1 t + \lambda_{vco}). \qquad (7)$$

The multiplier forms the product $v_0 v_1$, and this is

$$v_2 = \frac{A}{2}\left[\frac{2N}{A} f(t) \cos(\omega_1 t + \lambda_{vco}) + \sin(\lambda_i - \lambda_{vco})\right]. \qquad (8)$$

Here again, periodic terms with frequency $2\omega_1$ have been ignored. In practice, the vco does not respond to these terms when they appear at its input. The amplifier output is

$$v_4 = Gv_2 \qquad (9)$$

and because v_4 controls the frequency of the voltage-controlled oscillator,

$$\dot{\lambda}_{vco} = v_4 \times \left(\frac{1}{\text{volt-second}}\right). \qquad (10)$$

Eqs. (8)-(10) combine to give

$$\dot{\lambda} + K \sin \lambda = KJf(t) \cos(\omega_1 t + \lambda_i + \lambda) \qquad (11)$$

where $K = \frac{1}{2}GA$, $J = 2N/A$, and λ has been written for $\lambda_{vco} - \lambda_i$. Eq. (11) describes the behavior of the simple loop without any approximations other than those implicit in the use of an ideal multiplier to represent a physical component.

In cases of practical interest the phase modulation λ is small—certainly less than $\pm 90°$. In addition, if the statistical properties of the time function $f(t)$ are fixed, the amplitude of λ must increase if N, which may be regarded as the amplitude of the noise, is increased. Thus if λ is written as a power series[7]

$$\lambda(t) = \lambda_0(t) + J\lambda_1(t) + J^2\lambda_2(t) + J^3\lambda_3(t) + \cdots \qquad (12)$$

the solution of (11) can be reduced to solving for the coefficients $\lambda_0(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, $\cdots$ each of which is a random time function. This is easily done by substituting the series (12) into (11) and replacing $\sin \lambda$ by $(\lambda - \lambda^3/6 + \cdots)$ and $\cos \lambda$ by $(1 - \lambda^2/2 + \cdots)$ whenever they appear and then equating the coefficients of equal powers of J. The result is the series of linear equations,

$$\lambda_0 = 0 \qquad (13)$$

$$\dot{\lambda}_1 + K\lambda_1 = Kf(t) \cos(\omega_1 t + \lambda_i) \qquad (13a)$$

$$\dot{\lambda}_2 + K\lambda_2 = K[-\lambda_1 f(t) \sin(\omega_1 t + \lambda_i)] \qquad (13b)$$

$$\dot{\lambda}_3 + K\lambda_3 \qquad (13c)$$

$$= K\left[\frac{\lambda_1^3}{6} - \frac{\lambda_1^2}{2} f(t) \cos(\omega_1 t + \lambda_i) - \lambda_2 f(t) \sin(\omega_1 t + \lambda_i)\right]$$

$$\dot{\lambda}_n + K\lambda_n = K[\text{a function of } \lambda_{n-1}, \lambda_{n-2}, \cdots, \lambda_1].$$

These equations have a simple interpretation. Eq. (13a) together with the series (12) implies that a first approximation to the effect of noise on the loop can be found by analyzing the system shown schematically in Fig. 3(a). The noise $f(t)$ is multiplied by $\cos(\omega_1 t + \lambda_i)$ and the resulting low-frequency noise after passing through a low-pass filter with transfer function $K/(S + K)$ and an attenuator with a transmission equal to J gives the phase modulation $\lambda(t)$.

Fig. 3(b) is a schematic representation which includes the effect of the first three terms in the series (12). An even more detailed physical model can be constructed by including four terms in the series (12) as shown in Fig. 3(c). Despite the seeming complexity of these models, the computation is straightforward in that there is no feedback; all signals flow from left to right. In addition, each equation contains only one unknown.

The physical model shown in Fig. 3(b) is useful in finding
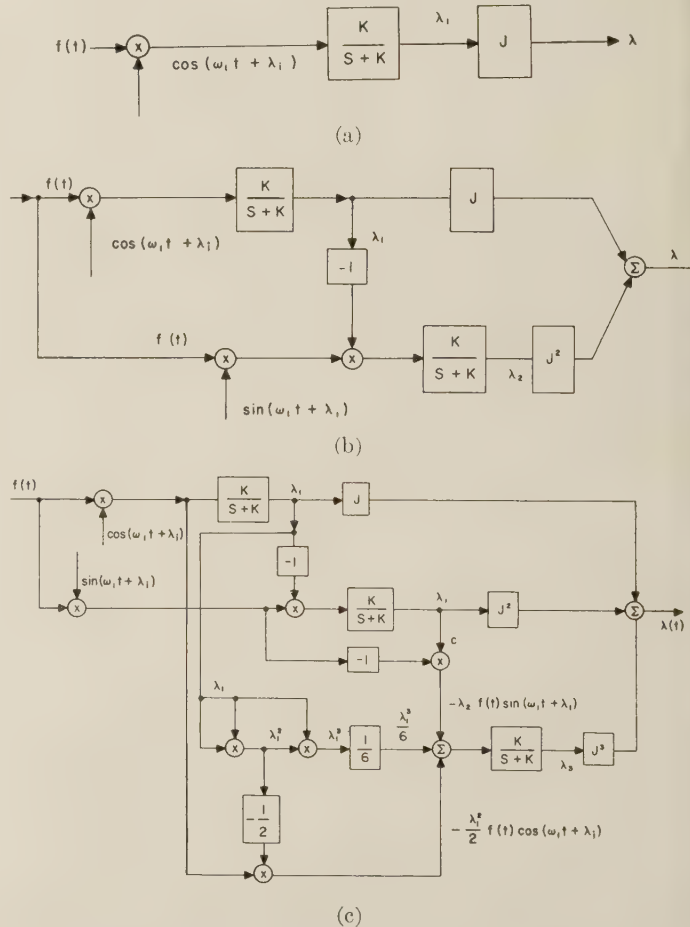

(a)

(b)

(c)

Fig. 3—(a) First approximation to effects of noise on a simple loop. (b) Model for second approximation to effects of noise on a simple loop. (c) Model for third approximation to effects of noise on a simple loop.

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[7] The use of this series expansion was suggested by L. R. Welch of the Jet Propulsion Lab., Pasadena, Calif.

ly steady-state phase error which may be produced when the noise $Nf(t)$ is multiplied by the phase-modulated vco output. Any correlation between these waveforms would be expected to produce a dc component in the output of the multiplier which would eventually manifest itself as a steady-state phase offset. The physical model makes it clear that λ_1 contains no steady component, but that λ_2 may contribute a steady-state phase error owing to the multiplication of the time function $f(t) \sin(\omega_1 t + \lambda_i)$ by λ_1, which is just a filtered version of $f(t) \cos(\omega_1 t + \lambda_i)$. Since the impulse response of a system with transfer function $K/(S + K)$ is $K\epsilon^{-K\tau}$ the time function λ_1 is the convolution of $K\epsilon^{-K\tau}$ and $f(t) \cos(\omega_1 t + \lambda_i)$, i.e.,

$$\lambda_1 = \int_0^\infty K\epsilon^{-K\tau} f(t-\tau) \cos[\omega_1(t-\tau) + \lambda_i]\, d\tau \qquad (14)$$

and the average value of λ_2 is just the average value of $-\lambda_1 f(t) \sin(\omega_1 t + \lambda_i)$ or

$$\lambda_{2\text{avg}} = -\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left\{ \int_0^\infty K\epsilon^{-K\tau} f(t-\tau) \right.$$

$$\left. \cdot \cos[\omega_1(t-\tau) + \lambda_i]\, d\tau \right\} f(t) \sin(\omega_1 t + \lambda_i)\, dt. \qquad (15)$$

Since $f(t)$ is assumed to be a random time function with no periodic components, (15) reduces to

$$\lambda_{2\text{avg}} = -\frac{K}{2} \int_0^\infty \epsilon^{-K\tau} \phi_{ff}(\tau) \sin \omega_1 \tau\, d\tau \qquad (16)$$

where ϕ_{ff} has its usual definition,

$$\phi_{ff} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) f(t-\tau)\, d\tau. \qquad (17)$$

For the specific case in which $f(t)$ is white noise which has been passed through a filter with an ideal rectangular band-pass characteristic, as shown in Fig. 4,
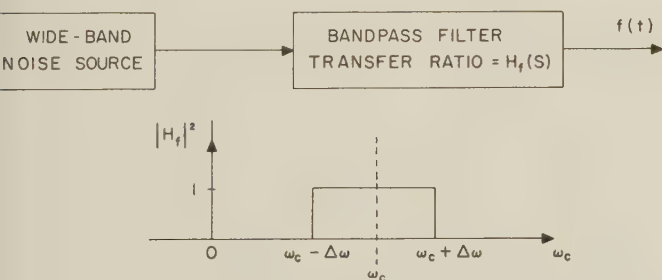

Fig. 4—Generation of $f(t)$.

$$\phi_{ff} = 4W \frac{\sin \Delta\omega\tau}{\tau} \cos \omega_c \tau \qquad (18)$$

$$= \sigma_f^2 \frac{\sin \Delta\omega\tau}{\Delta\omega\tau} \cos \omega_c \tau \qquad (18a)$$

here W is the noise power per unit bandwidth. For this type of noise,

$$\lambda_{\text{avg}} = -\frac{K\sigma_f^2}{2\Delta\omega} \int_0^\infty \frac{\epsilon^{-K\tau}}{\tau} \sin \Delta\omega\tau \cos \omega_c \tau \sin \omega_1 \tau\, d\tau. \qquad (19)$$

The definite integral (19) can be found in integral tables;[8] its value is

$$\lambda_{2\text{avg}} = -\frac{K\sigma_f^2}{16\Delta\omega} \ln \frac{K^2 + (\omega_1 + \omega_c + \Delta\omega)^2}{K^2 + (\omega_1 + \omega_c - \Delta\omega)^2}$$

$$- \frac{K\sigma_f^2}{16\Delta\omega} \ln \frac{K^2 + (\omega_1 - \omega_c + \Delta\omega)^2}{K^2 + (\omega_1 - \omega_c - \Delta\omega)^2}. \qquad (20)$$

If the noise is confined to a narrow band, $\omega_1 + \omega_c \gg \Delta\omega$. In this case, the first term in (20) becomes small compared to the second term; evaluation of the second term is simplified by the introduction of the normalized variables

$$\omega_0' = \frac{\omega_c - \omega_1}{K}$$

and

$$\Delta\omega' = \frac{\Delta\omega}{K}.$$

For narrow-band noise

$$\lambda_{2\text{avg}} \cong \frac{\sigma_f^2}{16\Delta\omega'} \ln \frac{1 + (\omega_0' + \Delta\omega')^2}{1 + (\omega_0' - \Delta\omega')^2} \qquad (21)$$

which is an odd function of ω_0'. The resulting phase error is

$$J^2 \lambda_{2\text{avg}} \cong \frac{1}{4} \frac{N^2\sigma_f^2}{A^2\Delta\omega'} \ln \frac{1 + (\omega_0' + \Delta\omega')^2}{1 + (\omega_0' - \Delta\omega')^2}. \qquad (22)$$

The ratio $N^2\sigma_f^2/A^2$ is [from (6) and (18)] just the square of the ratio of rms noise to peak signal, measured at the input to the loop. The phase bias is thus a function of: $N^2\sigma_f^2/A^2$, the ratio of mean-square noise to the square of signal amplitude; $\Delta\omega'$, the ratio of noise bandwidth (see Fig. 4) to loop bandwidth K; and ω_0', the ratio of ω_0 to loop bandwidth. Here ω_0 is the difference between the signal frequency ω_1 and the noise center frequency, ω_c. Fig. 5 is a plot of $J^2\lambda_2$ vs ω_0' for $N^2\sigma_f^2/A^2 = 1$ and $\Delta\omega' = 1$.

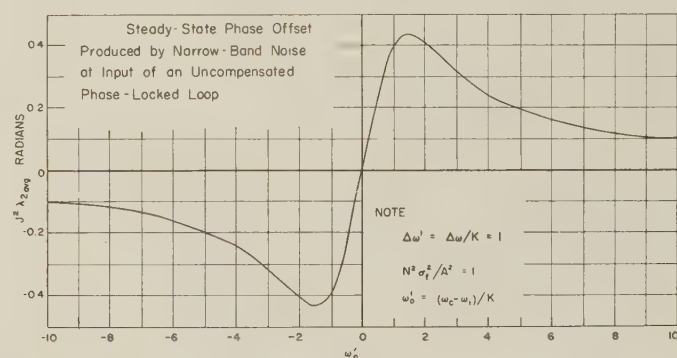Fig. 5.

The phase error vanishes when $\omega_0' = 0$; i.e., when the center-frequency of the noise band coincides with ω_1.

Noise in Loops with RC Compensation

The analysis of practical phase-lock circuits, which include an RC compensation network as shown in Fig. 2,

[8] W. Grobner and N. Hofreiter, "Integraltafel; Zweiter Teil, Bestimmte Integrale," Vienna, Springer-Verlag; 1950.

follows the same general plan as that used to analyze the uncompensated loop. As before, ignoring components in v_2 with frequency $2\omega_1$

$$v_1 = A \sin (\omega_1 t + \lambda_i) + N f(t) \qquad (23)$$

$$v_0 = \cos (\omega_1 t + \lambda_{\text{vco}}) \qquad (24)$$

$$v_2 \cong \frac{A}{2} \left[\frac{2N}{A} f(t) \cos (\omega_1 t + \lambda_{\text{vco}}) + \sin (\lambda_i - \lambda_{\text{vco}}) \right] \qquad (25)$$

and

$$\dot{\lambda} = v_4 \times \left(\frac{1}{\text{volt-second}} \right). \qquad (26)$$

Because of the inclusion of the RC network, however,

$$v_4 + T_1 \dot{v}_4 = G(v_2 + T_2 \dot{v}_2) \qquad (27)$$

where $T_1 = (R_1 + R_2) C$ and $T_2 = R_2 C$. $\qquad (28)$

Eq. (27) becomes

$$\ddot{\lambda} + \frac{1}{T_1} \dot{\lambda} = \frac{K}{T_1} \left\{ J f(t) \cos (\omega_1 t + \lambda_i + \lambda) - \sin \lambda \right.$$

$$\left. + T_2 \frac{d}{dt} [J f(t) \cos (\omega_1 t + \lambda_i + \lambda) - \sin \lambda] \right\}. \qquad (29)$$

Substitution of the power series (12) into (29) and collection of the coefficients of equal powers of J lead to a series of equations which resembles the series (13):

$$\lambda_0 = 0 \qquad (30)$$

$$\ddot{\lambda}_1 + \left(\frac{1}{T_1} + \frac{KT_2}{T_1} \right) \dot{\lambda}_1 + \frac{K}{T_1} \lambda_1 = \frac{K}{T_1} f(t) \cos (\omega_1 t + \lambda_i)$$

$$+ \frac{KT_2}{T_1} \frac{d}{dt} [f(t) \cos (\omega_1 t + \lambda_i)] \qquad (30a)$$

$$\ddot{\lambda}_2 + \left(\frac{1}{T_1} + \frac{KT_2}{T_1} \right) \dot{\lambda}_2 + \frac{K}{T_1} \lambda_2 = -\frac{K}{T_1} \lambda_1 f(t) \sin (\omega_1 t + \lambda_i)$$

$$- \frac{KT_2}{T_1} \frac{d}{dt} [\lambda_1 f(t) \sin (\omega_1 t + \lambda_i)] \qquad (30b)$$

$$\ddot{\lambda}_3 + \left(\frac{1}{T_1} + \frac{KT_2}{T_1} \right) \dot{\lambda}_3 + \frac{K}{T_1} \lambda_3 = \frac{K}{T_1} g(t) + \frac{KT_2}{T_1} \frac{d}{dt} g(t)$$

where

$$g(t) = \left[\frac{\lambda_1^3}{6} - \frac{\lambda_1^2}{2} f(t) \cos (\omega_1 t + \lambda_i) \right.$$

$$\left. - \lambda_2 f(t) \sin (\omega_1 t + \lambda_i) \right] \qquad (30c)$$

These equations permit the effect of noise to be interpreted in terms of a series of block diagrams. The interpretation of (30a) is shown in Fig. 6(a). A first approximation to the phase-modulation is found by multiplying the noise $f(t)$ by the unmodulated vco output $\cos (\omega_1 t + \lambda_i)$ and passing the resulting waveform through a filter with transfer function

$$H(S) = \frac{\dfrac{KT_2}{T_1} S + \dfrac{K}{T_1}}{S^2 + \dfrac{1}{T_1} + \dfrac{KT_2}{T_1} S + \dfrac{K}{T_1}}$$
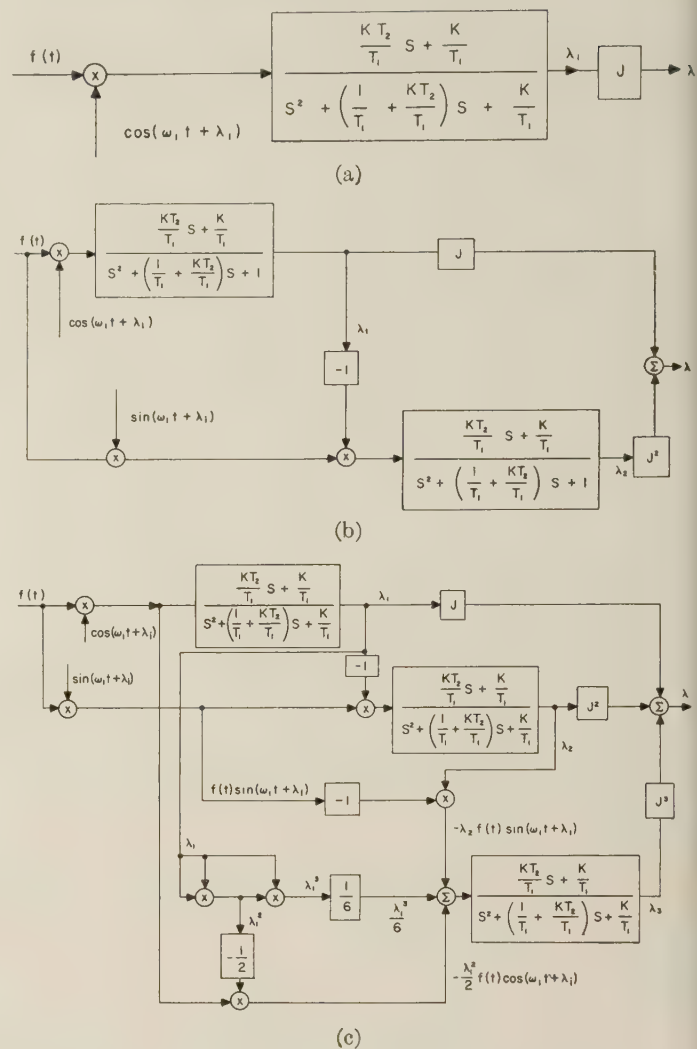
Fig. 6—(a) Model for first approximation to effects of noise on a compensated loop. (b) Model for second approximation to effects of noise on a compensated loop. (c) Model for third approximation to effects of noise on compensated loop.

and an attenuator with transmission J. More detailed models based on (30b) and (30c) are shown in Figs. 6(b) and 6(c). As before, all signals flow from left to right; there is no feedback.

THE USE OF A PHASE-LOCKED LOOP IN THE REFERENCE CHANNEL OF AN ANGLE-MEASURING SYSTEM

Fig. 7 is a simplified block diagram of a simultaneous-lobing angle-measuring system used to measure the azimuth angle of a remote signal source. Synchronous detection is used to discriminate against noise which may be produced by a second remote source. The relatively noise-free reference signal required by the synchronous detector is provided by a phase-locked loop.

The reference channel is assumed to be fed by a nondirectional antenna. On the other hand, the measurement channel is fed by an antenna designed to produce an output A_m proportional to the azimuth angle θ_s, with a phase reversal when the azimuth passes through zero, as shown in Fig. 8. The input from the antenna to the reference channel is

Fig. 7—Phase-locked loop used to provide reference channel for angle-measuring system.

$$v_1 = A_1 \sin (\omega_1 t + \lambda_i) + N_1 f(t) \qquad (31)$$

and the antenna and reference inputs to the measurement channel are

$$v_m = A_m(\theta_s) \sin (\omega_1 t + \lambda_i) + N_m(\theta_N) f(t) \qquad (32)$$

$$v_r = \sin (\omega_1 t + \lambda_{vco}). \qquad (33)$$

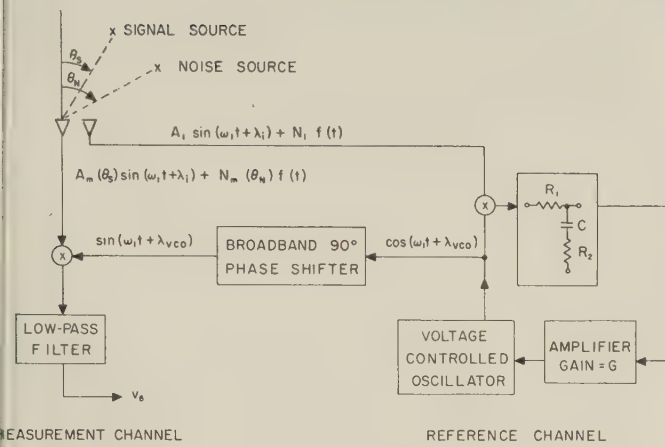In the absence of noise, $(N_1 = N_m = 0)$ the steady-state output of the vco will be $\cos (\omega_1 t + \lambda_i)$. The reference input to the angle-measuring multiplier will then be $\sin (\omega_1 t + \lambda_i)$, making the dc component of the multiplier
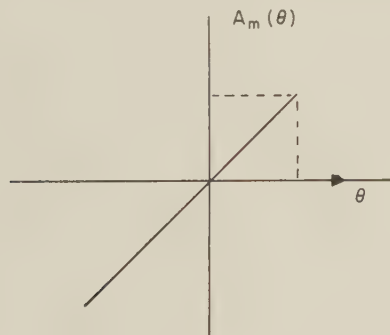

Fig. 8—Typical form for $A_m(\theta)$.

output just $A_m/2$. The factor A_m has the form given in Fig. 8, so that the output of the multiplier measures the angle of arrival of the signal $\sin (\omega_1 t + \lambda_i)$ provided there is no noise. Noise at the input to the reference channel produces a phase-jitter of the vco output. If this phase jitter results in a component in the vco output which is correlated with the noise in the measurement channel, v_5 will contain a spurious dc component owing to the noise. The results of the previous sections will be used to compute this spurious component.

To find the output of the angle-measuring multiplier when noise is present at the inputs of both channels, the product $v_r v_m$ is formed and is expanded in a power series in J. In this application both J and λ are defined for the reference channel by $J = 2N_1/A_1$ and $\lambda = \lambda_{vco} - \lambda_i$. When the coefficients of J are collected in the resulting

expression for the output of the angle-measuring multiplier, the result is

$$v_5 = \frac{A_m(\theta)}{2} + N_m f(t) \sin (\omega_1 t + \lambda_i)$$
$$+ J N_m \lambda_1 f(t) \cos (\omega_1 t + \lambda_i). \qquad (34)$$

The multiplier is assumed to contain a filter which rejects periodic components at the frequency $2\omega_1$. The first term in (34) represents the desired output. The second term has no dc component and can be made as small as desired by low-pass filtering of v_5. The term $J N_m \lambda_1 f(t) \cos (\omega_1 t + \lambda_i)$ contributes a dc error; the dc arises from the multiplication of λ_1, a filtered version of $f(t) \cos (\omega_1 t + \lambda_i)$, by $f(t) \cos (\omega_1 t + \lambda_i)$ itself. Thus the steady-state error in v_5 is

$$v_{5avg} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left\{ J N_m \int_0^\infty h(\tau) f(t - \tau) \right.$$
$$\left. \cdot \cos [\omega_1(t - \tau) + \lambda_i] \, d\tau \right\} f(t) \cos (\omega_1 t + \lambda_i) \, dt \qquad (35)$$

and if $f(t)$ contains no periodic components,

$$v_{5avg} = \frac{J N_m}{2} \int_0^\infty \phi_{ff}(\tau) h(\tau) \cos \omega_1 \tau \, d\tau \qquad (36)$$

where $h(\tau)$ is the inverse transform of $H(S)$ and ϕ_{ff} is the autocorrelation of $f(t)$. For noise with an autocorrelation function

$$\phi_{ff}(\tau) = \sigma_f^2 \frac{\sin \Delta\omega\tau}{\Delta\omega\tau} \cos \omega_1 \tau \qquad (37)$$

(produced by passing white noise through an ideal band-pass filter of bandwidth $2\Delta\omega$ centered at ω_1)

$$v_{5avg} = \frac{J N_m}{4} \sigma_f^2 \int_0^\infty \frac{\sin \Delta\omega\tau}{\Delta\omega\tau} (1 + \cos 2\omega_1\tau) h(\tau) \, d\tau. \qquad (38)$$

If $H(S)$ has the form given by (5) with parameters chosen for good response to transient charges in input phase, i.e.,

$$H(S) = \frac{\sqrt{2}\omega_N S + \omega_N^2}{S^2 + \sqrt{2}\omega_N S + \omega_N^2} \qquad (39)$$

the value of $h(\tau)$ is

$$h(\tau) = \sqrt{2}\omega_N \epsilon^{-(\sqrt{2}/2) \omega_N \tau} \cos \frac{\sqrt{2}}{2} \omega_N \tau \qquad (40)$$

using this specific form for $h(\tau)$ in (36) gives

$$v_{5avg} = \frac{J N_m \sigma_f^2}{4\Delta\omega} \sqrt{2}\omega_N$$
$$\cdot \int_0^\infty \frac{\sin \Delta\omega\tau}{\tau} \epsilon^{-(\sqrt{2}/2) \omega_N \tau} \cos \frac{\sqrt{2}}{2} \omega_N \tau \, d\tau$$
$$+ \frac{J N_m \sigma_f^2}{4\Delta\omega} \sqrt{2}\omega_N$$
$$\cdot \int_0^\infty \frac{\sin \Delta\omega\tau}{\tau} \epsilon^{-(\sqrt{2}/2) \omega_N \tau} \cos \frac{\sqrt{2}}{2} \omega_N \tau \cos 2\omega_1\tau \, d\tau. \qquad (41)$$

In the practical case where the noise bandwidth $\Delta\omega$ and the loop bandwidth ω_N are small compared to the carrier

frequency ω_1, the second term may be neglected in comparison to the first term. Using the known value of the first term gives[9]

$$v_{5\text{avg}} = \frac{\sqrt{2}}{2} \frac{\sigma_1}{A_1} \sigma_m \frac{\omega_N}{\Delta\omega} \left[\tfrac{1}{2}\arctan \frac{\sqrt{2}\,\dfrac{\Delta\omega}{\omega_N}}{1 - \left(\dfrac{\Delta\omega}{\omega_N}\right)^2} \right] \qquad (42)$$

when

$$\frac{\Delta\omega}{\omega_N} \leq 1$$

$$v_{5\text{avg}} = \frac{\sqrt{2}}{2} \frac{\sigma_1}{A_1} \sigma_m \frac{\omega_N}{\Delta\omega} \left[\frac{\pi}{2} + \tfrac{1}{2}\arctan \frac{\sqrt{2}\,\dfrac{\Delta\omega}{\omega_N}}{1 - \left(\dfrac{\Delta\omega}{\omega_N}\right)^2} \right] \qquad (42a)$$

when

$$\frac{\Delta\omega}{\omega_N} \geq 1.$$

For $\Delta\omega \ll \omega_N$ (noise bandwidth much less than loop bandwidth)

$$v_{5\text{avg}} \doteq \frac{1}{2} \frac{\sigma_1}{A_1} \sigma_m \qquad (43)$$

and for $\Delta\omega \gg \omega_N$ (noise bandwidth much greater than loop bandwidth)

$$v_{5\text{avg}} \doteq \frac{\sqrt{2}}{2} \frac{\pi}{2} \frac{\sigma_1}{A_1} \sigma_m \frac{\omega_N}{\Delta\omega} \qquad (44)$$

where $\sigma_1 = N_1\sigma_f$ and $\sigma_m = N_m\sigma_f$ are the rms noises in reference and measuring channels, respectively. The error in the measurement of the null point in the antenna pattern produced by the presence of correlated noise at the inputs of both the reference and measuring channels is thus a function of: σ_1/A_1, the ratio of rms noise to peak

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[9] The principal branch of the arctan function, running from arctan $-\infty = -\pi/2$ through arctan $0 = 0$ to arctan $+\infty = \pi/2$ should be used in evaluating (42) and (42a).
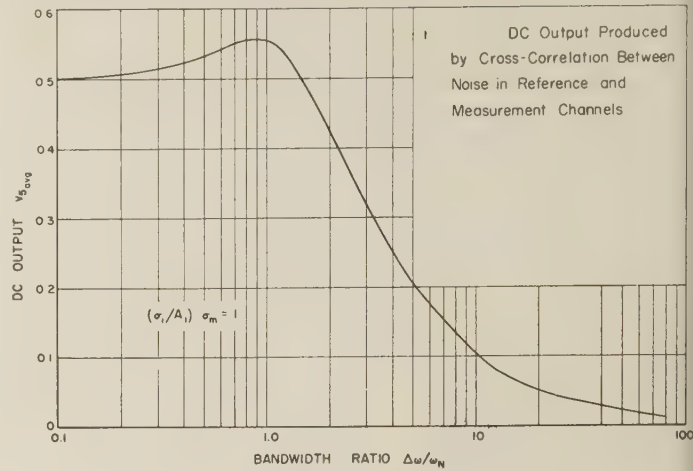
Fig. 9.

signal in the reference channel; σ_m, the rms noise in the measurement channel; and $\omega_N/\Delta\omega$, the ratio of loop bandwidth to noise bandwidth. A plot of $v_{2\text{avg}}$ vs $\Delta\omega/\omega_N$ for $(\sigma_1/A_1)\sigma_m = 1$ is given in Fig. 9. The voltage $v_{5\text{avg}}$ is converted to a phase error by dividing it by the calibration constant of the measurement channel, K_A volts-per-angular mil.

CONCLUSION

A method has been derived for determining the phase modulation produced in the output of a phase-locked loop by noise at its input. The method has been shown to be useful in predicting the steady-state phase offset produced by cross-correlation effects within a single loop and in predicting the errors which may occur when the loop is used to provide a reference channel for an angle-measuring system. The analysis presented in this paper permits the phase modulation resulting from input noise to be computed to any desired accuracy (at the cost of increasing analytic difficulty); a physical interpretation has been included as a guide to choice of a model sufficiently complex to explain the phenomenon sought, yet sufficiently simple to permit a solution to be found.

A Note on the Sampling Principle for
Continuous Signals*

A. V. BALAKRISHNAN†

Summary—Two sampling (integral interpolation) theorems for continuous signals (continuous parameter stochastic processes) are proved. The first of these is the sampling principle introduced by Shannon, precise formulation or proof of which has not appeared hitherto. Obtained as a secondary result in this connection is a generalization of a result on the spectra of sampled signals given by Bennet. The second theorem is a stochastic version of the Newton-Gauss interpolation formula as representative of a different class of sampling theorems.

IN THIS NOTE we state and prove two "sampling" theorems for continuous signals (continuous parameter stochastic processes). The first of these is the sampling principle of Shannon[1] and although it plays a fundamental role in the information theory of continuous signals, no precise (stochastic) formulation or proof has appeared in the literature. The second is a stochastic version of the Gauss-Newton interpolation formula for nonrandom functions.

THE SAMPLING PRINCIPLE OF SHANNON

Theorem 1

Let $x(t)^2 - \infty < t < \infty$, be a real or complex valued stochastic process, stationary in the "wide sense"[3] (or "second order" stationary) possessing a spectral density which vanishes outside the interval of angular frequency $[-2\pi W, 2\pi W]$. Then $x(t)$ has the representation:

$$x(t) = \lim_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)} \qquad (1)$$

for every t, where lim stands for limit in the mean square sense.[4]

Proof

The essence of the proof is that the right side of (1) is the best linear estimate, in the mean square sense, of $x(t)$ in terms of $\{x(n \mid 2W)\}$, with zero estimation error. The sampling principle per se for nonrandom functions

is applied to the covariance function[3] $R(t)$ of the process to yield:

$$R(t - \tau) = \sum_{-\infty}^{\infty} R\left(\frac{n}{2W} - \tau\right) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}. \qquad (2)$$

Thus, let $x^+(t)$ be the best linear estimate of $x(t)$ in terms of $\{x(n/2W)\}$ in the mean square sense. Then

$$x^+(t) = \lim_{-\infty}^{\infty} x\left(\frac{n}{2W}\right) a_n(t)$$

where the $a_n(t)$ satisfy the discrete version of the Wiener-Hopf equation:

$$R\left(t - \frac{m}{2W}\right) = \sum_{-\infty}^{\infty} R\left(\frac{n - m}{2W}\right) a_n(t) \qquad (3)$$

for every m and every t. We note that the series need not converge absolutely. Since $R(t)$ is the Fourier transform of the spectral density known to vanish outside $[-2\pi W, +2\pi W]$, it follows from the "sampling" principle commonly attributed to E. T. Whittaker[5] (integral interpolation) that (3) is satisfied with

$$a_n(t) = \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}.$$

We shall actually prove (2) here, since the oft-quoted proof given by J. M. Whittaker[6] leaves room for doubt as to whether or not $R(t)$ has to be further restricted for (2) to hold. Thus, let $\varphi(f)$ be the spectral density of the process so that

$$R(t - \tau) = \int_{-W}^{W} e^{2\pi i f(t-\tau)} \varphi(f) \, df.$$

Now, $e^{2\pi i f t}$ as a function of f can be expanded in a Fourier series in $[-W, +W]$ to yield:

$$e^{2\pi i f t} = \sum_{-\infty}^{\infty} a_n e^{in\pi f/W} \qquad (4)$$

for every f in the open interval $(-W, W)$ where the Fourier coefficients $\{a_n\}$ can be expressed:

$$a_n = \frac{1}{2W} \int_{-W}^{W} e^{2\pi i f t} e^{-i\pi nf/W} \, df$$

$$= \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}.$$

[1] C. E. Shannon, "A mathematical theory of communication," *Bell Sys. Tech. J.*; August, 1948.
[2] Throughout this note we assume all processes have finite variances and means. Further, all equalities involving random variables are understood to be with probability one.
[3] J. L. Dobb, "Stochastic Processes," John Wiley and Sons, Inc., New York, N. Y.; 1953.
[4] More explicitly this means.

$$\lim_{-\infty} E\left\{\left[\left| x(t) - \sum_{-N}^{N} x\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt - n)}{\pi(2W_t - n)} \right| \right]^2\right\} = 0.$$

As is known, this implies convergence in probability as well.

[5] E. T. Whittaker, "On the functions which are represented by the expansions of the interpolation theory," *Proc. Roy. Soc., Edinburgh*, vol. 35; 1915.
[6] J. M. Whittaker, "Interpolatory Function Theory," Cambridge University Press, Cambridge, Eng.; 1935.

The important fact about (4) is that since $e^{2\pi i f t}$ is absolutely continuous in $(-W, +W)$, it follows[7] that the series in (4) converges boundedly to $e^{2\pi i f t}$ in the open interval $(-W, -W)$. Hence by the Lesbesgue convergence theorem,[8] we obtain

$$R(t - \tau) = \lim_{N \to \infty} \int_{-W}^{W} \sum_{-N}^{N} a_n e^{i\pi n f/W} e^{-2\pi i f \tau} \varphi(f) \, df$$

$$= \lim_{N \to \infty} \sum_{-N}^{N} R\left(\frac{n}{2W} - \tau\right) a_n \qquad (5)$$

since

$$R\left(\frac{n}{2W} - \tau\right) = \int_{-W}^{W} e^{i\pi n f/W} e^{-2\pi i f \tau} \varphi(f) \, df$$

establishing (2) and, in particular, (3), as required.

We next prove that the error in estimation is zero. Since $x^+(t)$ is the optimal estimate, the mean square error:

$$E[|\, x(t) - x^+(t)\,|^2]$$

$$= E[(x(t) - x^+(t))(\overline{x(t)})]$$

$$= R(0) - E[x^+(t)\overline{x(t)}]$$

$$= R(0) - \sum_{-\infty}^{\infty} R\left(\frac{n}{2W} - t\right) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}$$

$$= 0,$$

using (3). This concludes the proof of the theorem.

As far as most applications are concerned, Theorem 1 is probably adequate. It is possible, however, to relax the requirement that the process have a spectral density.

Theorem 2

Let $x(t)$, $-\infty < t < \infty$, be a real or complex-valued stochastic process, second order stationary, having a spectral distribution $\Phi(f)$ such that

1) $$\int_{-\infty}^{-W} + \int_{W}^{\infty} d\Phi(f) = 0,$$

2) $\Phi(f)$ is continuous at $\pm W$.

Then $x(t)$ has again the representation (1).

Proof

If we proceed as in Theorem 1, we have only to establish (2) again. However, since $\Phi(f)$ is continuous at $\pm W$, it follows that the series in (4) converges boundedly almost everywhere with respect to the Lesbesgue-Stieltjes measure induced by $\Phi(f)$, so that the Lesbegue convergence theorem can be invoked again to obtain (5).

It is easy to demonstrate that the theorem is false if $\Phi(f)$ has a jump at $\pm W$. We have only to take

$$x(t) = \cos (2\pi W t + \theta)$$

[7] E. C. Titchmarsh, "Theory of Functions," Oxford University Press, Oxford, Eng., p. 408; 1950.
[8] *Ibid.*, p. 345.

where θ is a random variable with uniform probability density in $0 \le \theta \le 2\pi$. For, the right side of (1) in this case is

$$\lim \sum_{-\infty}^{\infty} x\left(\frac{n}{2W}\right) a_n = \left[\sum_{-\infty}^{\infty} a_n \cos n\pi \right] \cos \theta$$

$$= [\cos 2\pi W t] \cos \theta$$

$$\ne x(t) \quad \text{for} \quad t \ne \frac{n}{2W}.$$

The best we can state for the case where the spectral distribution has a jump at one or both end points is given below.

Corollary

Let $x(t)$ be a real or complex-valued second-order stationary stochastic process having a spectral distribution $\Phi(f)$ such that for some $W_0 > 0$,

$$\Phi(+\infty) - \Phi(W_0^+) + \Phi(-W_0^-) - \Phi(-\infty) = 0.$$

Then the mean squared error in the representation of $x(t)$ as

$$\lim \sum_{-\infty}^{\infty} x\left(\frac{n}{2W_0}\right) \frac{\mathrm{Sin}\, \pi(2W_0 t - n)}{\pi(2W_0 t - n)}$$

is given by

$$R(0)[(\text{jump of } \Phi(f) \text{ at } +W_0)$$

$$+ (\text{jump of } \Phi(f) \text{ at } -W_0)] \sin^2 2\pi W_0 t.$$

However, for every $W > W_0$, (1) is valid again with zero error.

Proof

The proof is omitted since it hinges merely on the equality

$$\sum_{-\infty}^{\infty} a_n \cos n\pi = \cos (2\pi W t)$$

as far as the first part is concerned, and the second part follows from the theorem.

The inverse problem of obtaining a continuous signal from a discrete signal leads to a useful converse of Theorem 2 (and 1).

Theorem 3

Let x_n, $-\infty < n < \infty$, be a discrete parameter real or complex valued second-order stationary stochastic process having a spectral distribution which is continuous at $\pm \frac{1}{2}$. Let

$$x(t) = \lim \sum_{-\infty}^{\infty} x_n \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)} \qquad (1a)$$

for every t, $-\infty < t < \infty$. Then $x(t)$ is a second-order stationary process having a spectral distribution satisfying conditions 1) and 2) of Theorem 2.

Proof

It is best to use the spectral representation theorem[9]

$$x_n = \int_{-1/2}^{1/2} e^{2\pi i n\lambda}\, dy(\lambda)$$

where $y(\lambda)$ is a (suitable normalized) orthogonal process with

$$E[|\, dy(\lambda)\,|^2] = dG(\lambda)$$

where $G(\lambda)$ is the spectral distribution of the process. Setting

$$a_n = \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}$$

we first show that the indicated limit in (1a) exists. The Nth partial sum therein can be written:

$$\sum_{-N}^{N} a_n x_n = \int_{-1/2}^{1/2} \sum_{-N}^{N} a_n e^{2\pi i n\lambda}\, dy(\lambda)$$

$$= \int_{-W}^{W} \sum_{-N}^{N} a_n e^{\pi i n f/W}\, dy(f/2W).$$

Since $G(f/2W)$ has no jumps at $\pm\, W$, it follows that $\sum_{-N}^{N} a_n\, e^{\pi i n f/W}$ converges boundedly almost everywhere with respect to the measure $dG(f/2W)$ to $e^{2\pi i f t}$ for $-W \le f \le W$, and hence also in the mean square sense. The process $y(\lambda)$ being orthogonal, the mean square convergence of the series in (1a) follows from this. Further we have:

$$x(t) = \int_{-W}^{W} e^{2\pi i f t}\, dy(f/2W).$$

From the spectral representation theorem for continuous parameter processes it follows that $x(t)$ is second-order stationary and that the covariance function $R(t)$ of the process is given by:

$$R(t) = \int_{-W}^{W} e^{2\pi i f t}\, dG(f/2W).$$

Moreover $G(f/2W)$ satisfies conditions 1) and 2) of Theorem 2, and further, therefore, $R(t)$ satisfies (2). That the theorem is false if the x_n process has a spectral distribution with a jump at $\pm\, W$ is demonstrated by the following example. Let

$$x_n = \cos(n\pi + \theta)$$

where θ is a random variable with uniform probability density in $0 \le \theta \le 2\pi$. Then

$$\lim \sum_{-\infty}^{\infty} x_n \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)} = [\cos 2\pi Wt]\cos\theta$$

which is clearly not stationary.

By way of corollary to this theorem we shall obtain a generalization of a result derived by Bennet[10] by a different method.

[9] Doob, *op. cit.*, p. 481.
[10] W. R. Bennet, "Methods of solving noise problems," PROC. IRE, vol. 44, pp. 609-638; May, 1956.

Corollary 1

Let the discrete process x_n of the theorem be derived by sampling from a continuous parameter process $y(t)$ so that

$$x_n = y(n/2W_0), \qquad W_0 > 0$$

where the $y(t)$ process is second-order stationary whose spectral distribution $\Phi(f)$ has no jumps at $\pm\, 2\,KW_0$, $k = 1, 2, \cdots$. Then if

$$x(t) = \lim \sum_{-\infty}^{\infty} x_n \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}$$

the $x(t)$ process is second-order stationary, with covariance function $R(t)$ given by

$$R(t) = \int_{-W}^{W} e^{2\pi i f t}\, d\mu(f)$$

where

$$d\mu(f) = \sum_{K=-\infty}^{\infty} d\Phi\left(\frac{W_0 f}{W} + 2kW_0\right).$$

Proof

We first derive an expression for the spectral distribution of the x_n process. For this we note that

$$E[(x_{m+n}\overline{x_m})] = \int_{-\infty}^{\infty} e^{(2\pi i n f/2W_0)}\, d\Phi(f)$$

and since

$$e^{(2\pi i n f/2W_0)} = e^{(2\pi i n/2W_0)(f + 2KW_0)},$$

for every integer k, we can write

$$E[(x_{m+n}\overline{x_m})] = \int_{-W_0}^{W_0} e^{(2\pi i n f/2W_0)}\, d\psi(f) \tag{7}$$

where

$$d\psi(f) = \sum_{K=-\infty}^{\infty} d\Phi(f + 2KW_0),$$

the necessary convergence being easily verified. By a change of variable in (7) we have

$$E[(x_{m+n}\overline{x_m})] = \int_{-1/2}^{1/2} e^{2\pi i n\lambda}\, dG(\lambda)$$

where

$$dG(\lambda) = d\psi(2W_0\lambda).$$

Moreover, the x_n process satisfies the conditions of the theorem, so that the $x(t)$ process is second-order stationary. Further, the covariance function $R(t)$ is given by (6), so that

$$R(t) = \int_{-W}^{W} e^{2\pi i f t}\, dG(f/2W)$$

$$= \int_{-W}^{W} e^{2\pi i f t}\left[\sum_{-\infty}^{\infty} d\Phi\left(\frac{W_0 f}{W} + 2KW_0\right)\right]$$

proving the corollary.

Perhaps a more useful form of this result is obtained if we require that the $y(t)$ process have a spectral density.

Corollary 2

Let the $y(t)$ process of Corollary 1 have an absolutely continuous spectrum with density $\varphi(f)$. Then $x(t)$ as defined in Corollary 1 is second-order stationary without any additional conditions on $y(t)$, and further, has an absolutely continuous spectrum with density defined (almost everywhere with respect to Lesbesgue measure) by:

$$\sum_{-\infty}^{\infty} \frac{W_0}{W} \varphi\left(\frac{W_0 f}{W} + 2kW_0\right) \qquad -W \leq f \leq +W$$

$$0 \qquad \text{otherwise.}$$

Proof

The main step that remains in the proof is to show that

$$\int_{-W}^{f} \sum_{-\infty}^{\infty} \frac{W_0}{W} \varphi\left(\frac{W_0 f}{W} + 2KW_0\right) df$$

$$= \sum_{\infty}^{\infty} \int_{-\infty}^{f} \frac{W_0}{W} \varphi\left(\frac{W_0 f}{W} + 2KW_0\right) df$$

and this follows from the fact that each $\varphi(W_0 f / W + 2kW_0)$ is nonnegative, and standard Lesbesgue integration theory.

NEWTON-GAUSS SAMPLING THEOREM

Besides the Cardinal series just discussed, the well-known integral interpolation formulas of Gregory, Newton, Gauss, Sterling, and Bessel also can be shown to have stochastic analogs. Here, however, we shall consider only the Newton-Gauss formula[6] since, as in the Cardinal series, it begins with an intermediate value and uses differences on either side of this value.

Theorem 4

Let $x(t)$ be a real or complex valued stochastic process satisfying the conditions of Theorem 2. Then $x(t)$ has the representation:

$$x(t) = \lim \Bigg[x(0)$$

$$+ \left\{ \frac{t}{h} \Delta x(0) + \frac{t(t-h)}{2!h^2} \Delta^2 x(-h) \right\}$$

$$+ \left\{ \binom{t+h}{3} \frac{\Delta^3 x(-h)}{h^3} + \binom{t+h}{4} \Delta^4 x(-2h) \right\}$$

$$+ \left\{ \binom{t+2h}{5} \Delta^5 x(-2h) + \binom{t+2h}{6} \Delta^6 x(-3h) \right\}$$

$$+ \cdots \Bigg] \tag{8}$$

for every t, where,

$$h = \tfrac{1}{2}w$$

$$\Delta x(nh) = x(nh + h) - x(nh)$$

$$\Delta^{K+1} x(nh) = \Delta[\Delta^K x(nh)]$$

$$\binom{t+mh}{r} = \frac{(t+mh)(t+mh-h)\cdots(t+mh-rh+h)}{r!}.$$

Proof

The essential idea of the proof is again that the right side of (8) is the optimal linear estimate, with zero error of $x(t)$ in terms of the random variables of the form $\{\Delta^K x(-nh)\}$ which are themselves linear combinations of the random variables $\{x(nh)\}$. It is possible to shorten the proof by using Theorem 2. First we note that

$$E[\Delta x(-mh)\overline{x(nh)}] = R(-mh - h - nh) - R(-mh - nh)$$

$$= \Delta R(-mh - nh)$$

where $R(t)$ is the covariance function of the process. Further, by induction, we have

$$E[\Delta^K x(-mh)\overline{x(nh)}] = \Delta^K R(-mh - nh). \tag{9}$$

Now by Theorem 2, $x(t)$ for each t, belongs to the closure (in the mean square sense) of the linear space generated by the random variables $\{x(nh)\}$. Hence to establish (8), it is enough to show that, using (9), for every integer m

$$R(t - mh) = \Bigg[R(-mh)$$

$$+ \left\{ \frac{t}{h} \Delta R(-mh) + \frac{t(t-h)}{2!h^2} \Delta^2 R(-mh - h) \right\}$$

$$+ \left\{ \binom{t+h}{3} \Delta^3 R(-mh - h) + \binom{t+h}{4} \Delta^4 R(-mh - 2h) \right\}$$

$$+ \cdots \Bigg]. \tag{10}$$

However, by a result due to Steffenson-Ferrar-Whittaker,[11] the convergence of (2) implies (10). Since from Theorem 2, (2) is known to hold, this proves the Theorem.

[11] J. M. Whittaker, *op. cit.*, p. 262.

A Note on Some Statistics Concerning Typewritten or Printed Material*

SID DEUTSCH†

THIS NOTE discusses some of the statistics associated with a sample of typewritten or printed material. The sample is shown in Fig 1. In the vertical direction, the sample includes 5 rows of type and the equivalent of 5 spaces between the rows of type. The vertical lines of Fig. 1 approximately represent the scanning lines for facsimile transmission. Although the original sample consisted of printed material (m, for example, occupies much more space than i), the size of the letters is approximately equal to that of typewritten copy. Some minor differences between printed and typewritten copy will exist because typewritten letters fall underneath each other, so that the spaces between the letters tend to form continuous columns.

For statistical purposes, it is assumed that the sample is composed of discrete resolution elements that are square. Furthermore, it is assumed that the signal is quantized so that if the area of a square is over 50 per cent black, the entire square is black, etc. The quantized sample is shown in Fig. 2, and corresponds to the signal obtained with a sampling and binary quantization process.

The highest probabilities of black or white messages are listed in Table I.

A "zero" in the sequence represents a white resolution element, etc. If $P(B_i)$ is the probability with which sequence B_i occurs, then the entropy per resolution element using blocks of N symbols is given by

$$G_N = -\frac{1}{N}[P(B_1) \log_2 P(B_1) + P(B_2) \log_2 P(B_2) + \cdots]. \quad (1)$$

The sample being considered here contains 5025 resolution elements. In the limiting case, where $N = 5025$, G_{5025} is equal to H, the entropy per resolution element. As N is increased, G_N approaches H.

In counting blocks of symbols, the last box in each line was assumed to be grouped with the first box in the next line. This minimizes the effects of the borders on the sample.

A better approximation to H is given by the conditional entropy of the next symbol when the $(N - 1)$ preceding symbols are known. This is given by

$$F_N = NG_N - (N - 1)G_{N-1}. \quad (2)$$

TABLE I

Message	Direction of Scan	Length of Message, Boxes	Per Cent Probability
Black	Vertical	1	60.5
Black	Vertical	2	14.3
Black	Vertical	3	6.17
Black	Vertical	7	6.17
White	Vertical	1	12.8
White	Vertical	2	12.8
White	Vertical	8	14.1
Black	Horizontal	1	41.1
Black	Horizontal	2	39.4
Black	Horizontal	3	9.21
White	Horizontal	2	26.8
White	Horizontal	3	27.8
White	Horizontal	4	14.3

The entropy of a block of N symbols was also considered.[1] If $N = 2$, for example, the possible sequences are as follows: $B_1 = 00$, $B_2 = 01$, $B_3 = 10$ and $B_4 = 11$.

[1] C. E. Shannon and W. Weaver, "The Mathematical Theory of Communication," University of Illinois Press, Urbana, Ill., p. 25; 1949.

Fig. 1.

Fig. 2.

TABLE II

N	Vertical Scan		Horizontal Scan		Bidirectional Scan
	G_N	F_N	G_N	F_N	G_N
1	0.6844	0.6844	0.6844	0.6844	—
2	0.6234	0.5624	0.6442	0.6040	—
3	0.5990	0.5502	—	—	0.6107
7	0.5601	—	—	—	—

box long, $B_2 = 2$ boxes long, etc. Then $P(B_1)$ is the probability with which black messages 1 box long occur etc.; N, in this case, is unity, since each black length is treated as an individual message. If these changes are made, (1) gives the entropy per black message (or similarly, per white message.) The results here are given in Table III.

TABLE III

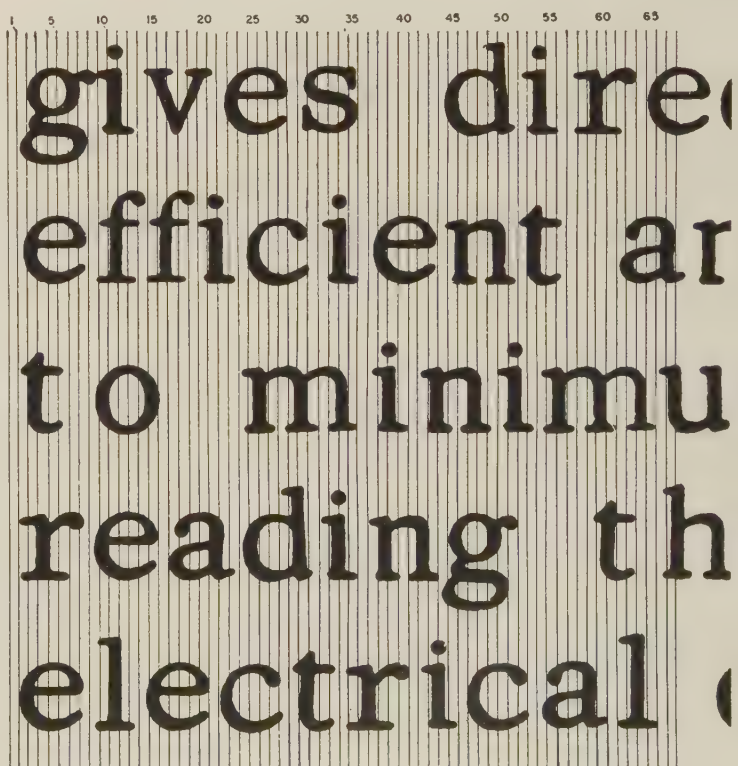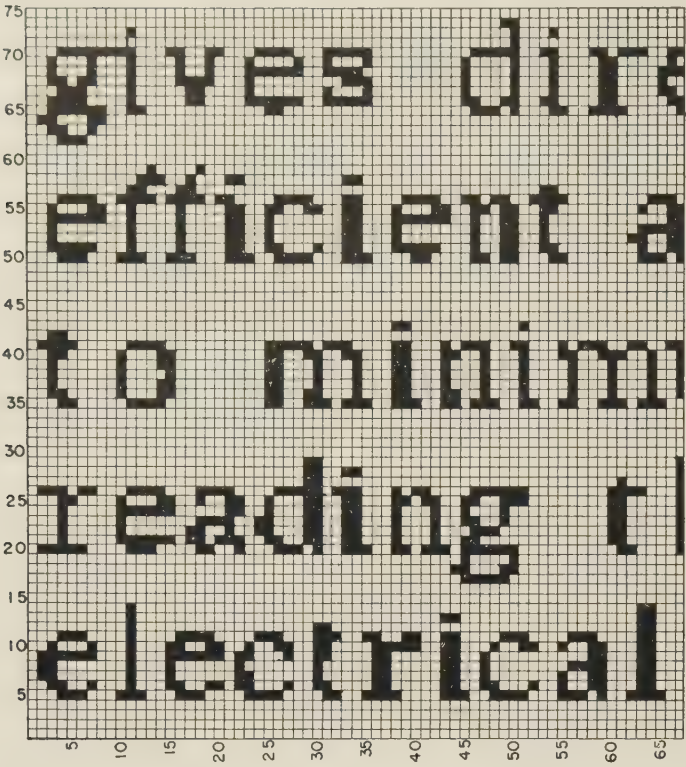| Message	Direction of Scan	H bits/message
Black	Vertical	2.002
White	Vertical	4.132
Black	Horizontal	1.871
White	Horizontal	3.170

A calculation was also made of G_3 for bidirectional scan; that is, each block of 3 symbols was arranged as follows:

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline & 3 \\ \hline \end{array}.$$
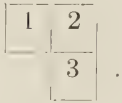
Numbers 1, 2, and 3 represent the first, second, and third digit of each possible sequence.

The various G_N and F_N values are summarized in Table II.

Eq. (1) gives the entropy per resolution element using blocks of N symbols. Suppose that the B_i symbols represent the various lengths of black, such as $B_1 = 1$

On an average, then, vertical scan requires 3.067 bits/message. Out of the total of 5025 boxes, there are 810 black and white messages. This corresponds to 6.203 boxes/message. Potentially, there is a time-bandwidth saving here of 6.203/3.067, or 2.022.

On an average, horizontal scan requires 2.52 bits/message. There are 934 horizontal black and white messages, or 5.38 boxes/message. Potentially, there is a time-bandwidth saving horizontally of 2.135.

One can conclude that a 2 to 1 saving is possible with typewritten or printed material. A code that could yield this much compression would be very complicated, but relatively simple codes should be capable of saving 1.5 to 1.

A Note on the Construction of a Multivariate Normal Sample*

G. MARSAGLIA

Summary—This note points out the superfluity of a method of [Stei]n and Storer for constructing a multivariate normal sample, and [sug]gests a simple alternative.

[S]TEIN and Storer[1] have recently discussed the problem of constructing samples having a specified multivariate normal distribution. They are [app]arently unaware of simple facts concerning the [beh]avior of the covariance matrix under a linear trans-[for]mation. If $\xi = (x_1, \cdots, x_n)$ is a $1 \times n$ vector whose [com]ponents are random variables having covariance [ma]trix C, and if M is an $m \times n$ matrix of constants, [the]n the covariance matrix of the components of ξM [is] $M'CM$. Thus, if $x_1, \cdots, x_n$ are independent, normally [dis]tributed, with variances 1, the components of ξM [wil]l be jointly normally distributed with a specified [cov]ariance matrix S if, and only if,

$$M'M = S. \tag{1}$$

[O]ne need not go to such extravagant lengths as the [ort]hogonal diagonalization of S in order to solve (1). [Var]ious well-known elementary procedures may be used. [Th]ere are exactly 2^n triangular (zeros below the diagonal) $[m] \times n$ matrices M which solve (1), for if S_k is the matrix

of elements common to the first k rows and columns of S, and if S_{k+1} is partitioned so:

$$S_{k+1} = \begin{pmatrix} S_k & \alpha'_k \\ \alpha_k & d_k \end{pmatrix},$$

then sequences $M_1, M_2, \cdots$ and $M_1^{-1}, M_2^{-1}, \cdots$ such that $M'_k M_k = S_k$ may be constructed by the relations

$$M_{k+1} = \begin{pmatrix} M_k & \beta'_k \\ 0 & b_k \end{pmatrix}, \qquad M_{k+1}^{-1} = \begin{pmatrix} M_k^{-1} & -b_k^{-1} M_k^{-1} \beta'_k \\ 0 & b_k^{-1} \end{pmatrix}$$

where $\beta_k = \alpha_k M_k^{-1}$ and $b_k^2 = d_k - \beta_k \beta'_k$. There are exactly two choices for b_k, since S_{k+1} is positive definite, and

$$(\beta_k M_k^{-1\prime}, \; -1) S_{k+1} (\beta_k M_k^{-1\prime}, \; -1)' = d_k - \beta_k \beta'_k > 0.$$

We apply this procedure to the case discussed.[1] If $e^{a|t|}$ is the covariance function of a stationary normal stochastic process y_t, and if $t_1 < t_2 < \cdots < t_n$ are "time" points, not necessarily equally spaced, the covariance matrix S of $y_{t_1}, \cdots, y_{t_n}$ will have i, jth element $e^{a|t_i - t_j|}$. A solution to (1) in this case has i, jth element

$$m_{ij} = \begin{cases} 0, & j < i \\ e^{a(t_j - t_i)}, & 1 = i \le j \\ e^{a(t_j - t_i)} [1 - e^{2a(t_i - t_{i-1})}]^{1/2}, & 1 < i \le j. \end{cases}$$

A Bibliography of Information Theory
(Communication Theory—Cybernetics)*
(*Second Supplement*)

F. LOUIS H. M. STUMPERS†

Contents

Abstracts: (381–1955) at the end of a line refers to Abstract 381 of Abstract and References, Proceedings of the IRE, 1955.

     (A 10726–1953) refers to Abstract 10726, *Physics Abstracts*, 1953.

     (B 28, 8298, 1954) refers to Abstract 8298, *Psychological Abstracts* 28, 1954.

     (C 974–1954) refers to *Mathematical Review*, p. 974, 1954.

     (D 775–1955) refers to Abstract 775, *Electrical Engineering*, 1955.

     (E 78696–1955) refers to Abstract 78696, *Annales des Télécommunications*, 1955.

     (No attempt has been made to list all available abstracts.)

I. GENERAL THEORY

Adler, F. P. "Minimum Energy Lost of an Observation," IRE Trans., Vol. IT-1 (September, 1953), pp. 28–32.

Angot, A. "La XI Assemblée Générale de l'Union Radio Scientifique Internationale, Commission VI." *Ondé Elect.* Vol. 35 (1955), pp. 620–624.

Anon. "Information Theory," *Wireless World*, Vol. 61 (1955), pp. 545–546.

Ayer, A. J. *et al* Studies in Communication, Contributed to the Communication Research Centre, University College, London. Martin, Secker, and Warburg, London, 1955; rev. in *Nature*, Vol. 177 (1956), pp. 445–446.

Bar-Hillel, Y. "An Examination of Information Theory," *Phil. Sci.*, Vol. 22 (1955), pp. 86–105.

Barnard, G. A. "Simple Proofs of Simple Cases of the Coding Theorem," *Proc. London Symposium* (1955), pp. 96–102.

Bayard, M. and Roquet, R. "La Caractérisation Arithmétique des Messages et son Utilisation pour Déterminer les Réseaux Correcteurs," *Ann. Télécomm.*, Vol. 11 (1956), pp. 33–45.

Bello, F. "The Information Theory," *Fortune* (December, 1953). *J. Telecomm. Aust.*, Vol. 10 (1954), pp. 5–13. (E74551).

Bending, A. W. "Transmitted Information and the Length of Rating Scales," *J. Exp. Psychol.*, Vol. 47 (1954), pp. 303–308. (B29, 1745).

Blachman, N. M. "The Third London Symposium on Information Theory," IRE Trans., Vol. IT-2 (March, 1956), pp. 17–23.

Blank, A. A. and Quastler, H. *Notes on the Estimation of Information Measures*, Tech. Rep. R-56, Control Syst. Lab. University of Illinois, 1954.

Blokh, E. L. and Harkevic, A. A. "Geometric Representations in Communications Theory," *Izv. Akad. Nauk. SSSR*, Ord. Tech. Nauk. no. 6 (1955), pp. 91–100. (In Russian.) (C51-52, 1956).

——. "Geometric Theory of the Threshold of Transmissibility of Communication Systems," *Radiotekhnika*, Vol. 10 (1955), pp. 3–7. (In Russian.) (D811, 1956).

Brillouin, L. "Science et Information," *Bull. Assoc. Anciens Eleves Ec. Polyt. Fr.*, Vol. 59 (1956), pp. 51–55.

Chang, S. S. L. "Theory of Information Feedback Systems," IRE Trans., Vol. IT-2 (September, 1956), pp. 29–40.

Cherry, E. C. (ed.). "Information Theory (Symposium London 1955)," p. 401, 1956. Butterworth Scientific Publishers.

——. "Information Theory, Third London Symposium," *Nature*, Vol. 176 (1955), pp. 773–774. (E78657).

Copegui, J. R. de. "La Medida de la Informacion," *Rev. Telecommunic. Esp.*, Vol. 9 (1955), pp. 14–22. (E80209).

Couffignal, L. "Science et Technique de l'Information (Essai Methodologique)," *2e Partie Struct. Evol. Tech.*, Vol. 7 (1956), pp. 1–13. (E86972).

Davis, H. and Trautman, D. L. "The Removal of Redundancy due to Intersymbol Interference," 1955 IRE Convention Record, Part 4, pp. 182–185.

Duffieux, P. M. "La Théorie de l'Information et le Monde en Expansion, "*Rev. Opt. Théor. Instrum.*, Vol. 35 (1956), pp. 96–102.

Edwards, R. F. "Information Theory. A Key to the Procurement of Reliable Parts," *Proc. Sec. Natl. Symp. Quality Control* (1956), pp. 53–57.

Elias, P. "Channel Capacity without Coding." Quart. Prog. Rep M.I.T. Res. Lab. Electronics, (October, 1956), pp. 90–93.

——. "Amount of Information Provided by a Particular Symbol," Quart. Prog. Rep. M.I.T. Res. Lab. Electronics, (October, 1956) pp. 93–95.

Enomoto, H. "Transmission Rate of a Communication Channel Subjected to Fading." *J. Inst. Telecomm. Eng.* (Japan), Vol. 36 (1953), pp. 686–689. (In Japanese.) (D2050, 1955).

Fairthorne, R. A. "Generating Functions of Number Languages," *Proc. Int. Congr. Math. (Amsterdam)*, (1954), pp. 336–337.

nstein, A. "Error Bounds in Noisy Channels without Memory," IRE Trans., Vol. IT-1 (March, 1955), pp. 13–14.

nstein, J. "Information Theory Aspects of Propagation through Time-Varying Media," *J. Appl. Phys.*, Vol. 26 (1955), pp. 19–229. (2103, 1955).

isher, H. "An Introduction to the Theory of Information," *Libr. Quart.*, Vol. 25 (1955), pp. 326–332. (B30, 4487, 1956).

gel, L. J. "Towards a Measure for Meaning." Proc. IRE, Vol. 43 (August, 1955), p. 1018.

——. "A Communication Theory Approach towards the Design of Aircraft Instrument Displays," 1955 IRE CONVENTION RECORD, Part 5, pp. 15–30.

ihauf, H. "Zum Stand der Informationstheorie," *Nachr. Techn.*, Vol. 6 (1955), pp. 433–445.

bor, D. "Optical Transmission," *Trans. London Symp.* (1955), pp. 26–33.

oberd, W. O. "Information Theory as applied to Communications," *Telecomm. J. Australia*, Vol. 9 (1953), pp. 203–207.

ildman, S. "Certain Aspects of Coherence, Modulation, and Selectivity in Information Transmission Systems," 1956 IRE CONVENTION RECORD, Part 4, pp. 113–125.

ood, I. J. *Some Terminology and Notation in Information Theory*, Monogr. No. 155R (November, 1955), 5 pp. (D382, 1956). *Proc. IEE*, Vol. 103C (1955), pp. 200–204.

——. "Rational Decisions," *J. Roy. Stat. Soc.*, B14 (1952), pp. 107–114. (C981–1955).

ant, D. A. "Information Theory and the Discrimination of Sequences in Stimulus Events," [In *Current Trends*, see Patton, (1953), pp. 18–46.] [B29, 59, 1955].

reen, P. E., Jr. "A Bibliography of Soviet Literature in Noise, Correlation and Information Theory," IRE Trans., Vol. IT-2 (June, 1956), pp. 91–94. (E88925).

ershberger, W. D. *Principles of Communication Circuits*, New York: Prentice Hall, 244 pp., 1955. Rev. N. Marchand, Proc. IRE, Vol. 43 (September, 1956), p. 1679.

ncin, A. Ya. "On the Basic Theorems of Information Theory," *Uspehi Mat. Nauk.* (*N.S.*), Vol. 11 (1956), pp. 17–75. (In Russian.) (C1098, 1956).

onda, N. "Improvement in Communication Efficiency during a Finite Interval of Time," *J. Inst. Elect. Comm. Eng.* (*Japan*). (In Japanese.) (D2917, 1955) (E75376).

uffman, D. A. "Information Conversation and Sequence Transducers," *Proc. Symp. Inf. Networks*, (1955), pp. 291–307. (C725, 1955).

——. "The Synthesis of Linear Sequential Coding Networks," *Proc. London Symp.*, (1955), pp. 77–95.

asolera, D. "Informazione e Codificazione," *Com. Elettronica Telev. Milano* (1954), pp. 1170–1176.

alaba, R. E. and Juncosa, M. L. "Linear Programming and Optimal Telecommunication Networks," Proc. IRE, Vol. 44 (September, 1956), p. 1874.

asahara, Y. "Representation of Communication-Channel Capacities by Means of Electrical Networks," *J. Inst. Elect. Comm. Eng.* (*Japan*), Vol. 37 (1954), pp. 791–794. (D2919, 1955).

elly, J. L., Jr. "A New Interpretation of Information Rate," *Bell Syst. Tech. J.*, Vol. 35 (1956), pp. 917–926. (E87869).

——. "A New Interpretation of Information Rate," IRE Trans., Vol. IT-2, (September, 1956), pp. 185–189. Abstr. Proc. IRE, Vol. 44 (1956), p. 1644.

harkevich, A. A. and Blokh, E. L. "On the Capacity of a Communication System," *Radiotehnika*, Vol. 10 (1955), pp. 14–20, (In Russian). (E74549).

——. *Essai d'une Théorie Generale des Communications*, Moskwa (1955) 268 pp. (E-L 3741).

lein, M. L., Williams, F. K., and Morgan, H. C. "Information Theory," *Instr. and Autom.*, Vol. 29 (1956), pp. 1519–1524. (E87868).

reer, J. "Definitions," IRE Trans., Vol. IT-1 (1955), p. 50.

ullback, S. "An Application of Information Theory to Multivariate Analysis," *Ann. Math. Stat.*, Vol. 27 (1956), pp. 122–146.

aloë, M. "Information, Sélection, Information Sélective. La Sécurité est-elle une Constante d'Univers?" *Ann. Télécomm.*, Vol. 10 (1955), pp. 31–38. (D3864, 1955).

ehan, F. W. "Telemetering and Information Theory," IRE Trans., Vol. TRC-2 (November, 1954), pp. 15–19. (E-76566).

eifer, M. and Blachman, N. M. "Communication Theory in Digital Systems," *Control Eng.*, Vol. 3 (1956), pp. 72–77.

MacKay, D. M. "Complementary Measures of Scientific Information Content," *Methodos*, Vol. 7 (1955), pp. 63–90.

——. "The Place of 'Meaning' in the Theory of Information," *Proc. London Symp.* (1955), pp. 215–225.

Mandelbrot, B. "Distribution de Willis-Yule Relative aux Nombres d'Espèces dans les Genres Biologiques," *C. R. Acad. Sci. Paris*, Vol. 242 (1956), pp. 2223–2226.

——. "Der Ingenieur als Stratege: Verhaltenstheorien," *Nachr. Fachber.*, Vol. III (1956), pp. 32–40.

——, and Schützenberger, M. P., *Décisions et Information* (forthcoming book).

——. "Théorie des Informations. Diagnostic et Transduction en l'Absence du Bruit," Inst. Henri Poincaré, University of Paris, 73 pp., 1955.

——. "Théorie de la Precorrection des Erreurs de Transmission," (Théorème de Feinstein) *Ann. Télécomm.*, Vol. 10 (1955), pp. 122–134.

Mayer, H. F. "Der Mensch und die Nachrichtentechnik," *ETZ*, A, Vol. 77 (1956), pp. 421–427. (E86974).

McGill, W. J. "Multivariate Information Transmission," *Psychometrika*, Vol. 19 (1954), pp. 97–116.

McMillan, B. "The Mathematics of Information Theory," 1955 IRE CONVENTION RECORD, Part 4, pp. 48–51.

——. a.o. *Current Trends in Information Theory*, Pittsburgh: University of Pittsburgh Press, 201 pp., 1954.

——. "Mathematical Aspects of Information Theory," (*Current Trends* 1–17) (B29, 834, 1955).

Meyer-Eppler, W. *Anwendung der Informationstheorie auf Impulsprobleme* (Imp. t. 40–60, See XI, Winckel).

——. "Anwendungen der Kommunikationsforschung auf lautsprachliche und typographische Probleme," *Sprachforum*, Vol. 1 (1955), pp. 70–77.

Miller, K. S. and Zadeh, L. A. "Signal Representation in Terms of Periodic Functions," *Bull. Amer. Math. Soc.*, Abstr. 498, Vol. 61 (1955), p. 300.

Miller, G. A. "Information and Memory," *Sci. Amer.* (August, 1956), pp. 42–46.

Mine, H. "Application de la Théorie de l'Information à l'étude d'un Système de Télécommunication Utilisant un Compresseur Extenseur Instantanée," *J. Inst. Elect. Commun. Eng.* (*Japan*), Vol. 38 (1955), pp. 775–781. (In Japanese.) (E87918).

Mitchell, F. H. and Whitehurst, R. N. "Further Remarks on the Odd Ball Problem as an Example in Information Theory," *J. Appl. Phys.*, Vol. 26 (1955), pp. 778–779. (E77768).

Mosch, A. D. du. "On the Average Uncertainty of a Continuous Probability Distribution," *Appl. Sci. Res. B*, Vol. 4 (1955), pp. 469–473. (C168, 1956).

Muroga, S. "Sur la Capacité de Transmission d'un Circuit dans une Bande de Fréquences Continue affectée de Bruit," *J. Inst. Elect. Comm. Eng.* (*Japan*), Vol. 30 (1955), pp. 961–964. (In Japanese.) (E87856).

Nakagone, Y. "Les Erreurs se Produisant dans le Téléimprimeur Arythmétique en cas de Mauvais Fonctionnement d'un Element de Départ ou d'un Element d'Arrêt," *J. Inst. Telecomm. Eng.* (*Japan*), Vol. 38 (1955), pp. 290–295. (E77910).

Neidhart, P. "Der Begriff des Wirkungsgrades in der Informationstheorie Elektr.," *Rdschau*, Vol. 10 (1956), pp. 15–19. (E82669).

Nejman, M. S. "Sur la Théorie Générale des Signaux et la Théorie Générale des Systemes Asservies," *Radiotehnika* (*SSSR*), Vol. 10 (1955), pp. 13–16. (E78656).

Okoda, S. "On the Information Invariant Abstracts," IRE Trans., Vol. IT-2 (June, 1956), p. 95. *Bull. Yamagata Univ.* (*Eng.*), Vol. 3 (1954), pp. 95–111.

——. and Huziki, S. "On the Intrinsic Substance and Metrization of Communication," *J. Inst. Elect. Comm. Eng.* (*Japan*), Vol. 204 (1940), pp. 47–147. (In Japanese.)

Oper. App. Lab. *Tables for Computing Informational Measures*, Air Force Cambridge Res. Inst., 38 pp.

Oshima, S. and Enomoto, H. "A Fading Compensation System," *J. Inst. Elect. Comm. Eng.* (*Japan*), Vol. 37 (1954), pp. 851–856. (In Japanese.) (E77864, D3466, 1955).

Passau, Ph. "Télétransmission de Données dans le Cadre de la Théorie de l'Information," *Bull. Assoc. Ing. Elect. Inst. Montefiore*, Vol. 67 (1954), pp. 513–525.

Patton, R. A. *Current Trends in Information Theory*, Pittsburgh: University of Pittsburgh Press. 188 pp., 1953. (B29, 838, 1955).

Pierce, J. R. *Electrons, Waves, and Messages*, 309 pp., 1956. New

York: Doubleday Co. Rev. Proc. IRE, Vol. 44 (1956), pp. 427–428.

Pinsker, M. S. "Quantity of Information in Gaussian Stationary Process," *Compt. Rend. Acad. Sci.* (*U.R.S.S.*), Vol. 99 (1954), pp. 213–216. (1488, 1955; C495, 1955).

Plummer, C. B. "More Words per Minute per Kilocycle," 1956 IRE Convention Record, Part 8, pp. 14–16.

Powers, K. H. "A Prediction Theory Approach to Information Rates," 1956 IRE Convention Record, Part 4, pp. 132–139.

Price, R. "Statistical Theory Applied to Communication through Multipath Disturbances," M.I.T. Res. Lab. Electronics, Tech. Rep. 266, p. 70. (C273, 1955).

Randow, T. von. "The Mathematical Valuation of Information," *Conv. Elettron. Telev., Milan*, (1954), pp. 1182–1186. (D4905, 1955).

Reichardt, W. "Binarschritte und gerichtete Impulse im Lichte der Informations theorie," *Hochf. u. Electroak.*, Vol. 64 (1955), pp. 76–83. (E82667).

Ridenour, L. N. (ed.). *Modern Physics for the Engineer*, New York: McGraw-Hill, 499 pp., 1954.

— —. "Computing Machines and the Processing of Information," (in *Modern Physics . . .*), pp. 455–490.

Rocha, L. F. "Aspectos de la Teoria de la Información," *Rev. Telegr. Electron.* (*Argent.*), Vol. 45 (1955), pp. 527–533. (E87867).

Rosenberg, P. "Information Theory and Electronic Photogrammetry," *Photogramm. Eng.* (1955), pp. 543–555.

Rothstein, J. "Information Theory and Quality Control," 1956 IRE Convention Record, Part 4, pp. 3–11.

Sacerdoti, G. "Valutazione della Quantita d'Informazione, fornita da una Machina calcolatrice," *Conv. Elettron. Telev. Milan* (1954), pp. 1187–1194. (D4908, 1955).

Sato, H. "On the Statistics of a Code Channel," *Rep. Univ. Electro-Commun.*, Vol. 6 (1954), pp. 21–25. (In Japanese). (C725, 1955).

Schäfer, O. "Anwendung der Statistischen Betrachtungsweise bei der Untersuchung von Uebertragungssystemen," *Regl. tech.*, Vol. 4 (1956), pp. 276–280.

Schouten, J. F. "Nachricht und Signal," *Nachr. Tech. Fachber.*, Vol. 6 (1956), pp. 1–2.

— —. "Ignorance, Knowledge and Information," *Proc. London Symp.* (1955), pp. 37–46.

Schützenberger, M. P. "Les Problèmes de Diagnostic et l'Axiomatique des Informations," *Rev. Gén. Sci. Pures Appl.*, Vol. 62 (1955), pp. 222–226. (C277, 1956).

— —. "Sur les Problèmes de Communications Métriques," *C. R. Acad. Sci. Paris*, Vol. 240 (1955), pp. 724–726.

— —. "Contributions aux Applications Statistiques de la Théorie de l'Information," *Publ. Inst. Stat. Univ.* (*Paris*), Vol. 3 (1954), pp. 1–117.

— —. "On Some Measures of Information Used in Statistics," *Proc. London Symp.* (1955), pp. 18–25.

Shannon, C. E. "The Zero Error Capacity of a Noisy Channel," IRE Trans., Vol. IT-2 (September, 1956), pp. 8–19. Abstr. Proc. IRE, Vol. 44 (September, 1956), p. 1643.

— —. "Some Geometric Results in Channel Capacity," *Nachr. Tech. Fachber.*, Vol. 6 (1956), pp. 13–15.

— —. "Geometrische Deutung einiger Ergebnisse bei der Berechnung der Kanalkapazität," *Nachr. Tech. Z.*, Vol. 10 (1957), pp. 1–13.

Silverman, R. A. "A discussion of Rates for Randomly Varying Channels, Including an Unsolved Case," Quart. Prog. Rep. M.I.T. Res. Lab. Electronics, (October, 1955), pp. 61–69.

— —. "On Binary Channels and their Cascades," IRE Trans. Vol. IT-1 (September, 1955), pp. 19–27.

Soest, J. L. van. "Some Consequences of the Finiteness of Information," *Proc. London Symp.* (1955), pp. 3–7.

— —. "Informatie en Communicatie," *Electro-techniek*, Vol. 34 (1956), pp. 1–5. (In Dutch).

Solomonoff, R. J. "Effect of Heisenberg's Principle on Channel Capacity," Proc. IRE, Vol. 43 (April, 1955), p. 484.

Somenzi, V. "Informazione e Teoria dell'Informazione," *Conv. Elettron. Telev.* (*Milan*), (1954), pp. 1197–1202.

Syski, R. "Analogies between the Congestion and Communication Theories," *A.T.E.J.*, Vol. 11 (1955), pp. 220–243. (D815, 1956) (E87040).

Takada, Y. "On the Mathematical Theory of Communication," *Jap. J. Psychol.*, Vol. 25 (1954), pp. 110–117. (B30, 5606, 1956).

Teichmann, H. "Nachrichtentechnik in neuer Sicht," *Fernmelde Praxis*, Vol. 33 (1956), pp. 465–471. (E86090).

Trucco, E. "A Note on the Information Content of Graphs," *Bul Math. Bioph.*, Vol. 18 (1956), pp. 129–136, 237–254.

Turin, G. L. "Communication through Noisy, Random-Multipath Channels," 1956 IRE Convention Record, Part 4, pp. 154–166

URSI. *Proceedings XIth General Assembly*, The Hague, 1954. Par 6, Comm. VI Information Theory (Subcomm. VIa), pp. 122–128 135–140.

Vallée, R. "Quelques Aspects de la Théorie de l'Information," *Rev. Gén. Electricité*, Vol. 63 (1954), pp. 698–763.

— —. "Un Aspect du Problème de l'Observation," *Methodos*, Vol. (1955), pp. 289–294.

Wiener, N. "What is Information Theory?," IRE Trans., Vol IT-2 (June, 1956), p. 48.

Wiesner, J. B. "Communication Theory and Transmission of In formation" (in Ridenour: *Modern Physics . . .* pp. 427–454).

II. a) BANDWIDTH AND TRANSMISSION CAPACITY, TIME FREQUENCY UNCERTAINTY; b) SIGNAL-TO-NOISE RATIO c) INSTANTANEOUS FREQUENCY; d) ANALYTICAL SIGNALS

a)

Bendat, J. S. "Optimum Time-Variable Filters for Finite Operatin Periods," *Bull. Amer. Math. Soc.* Abstr. 768, Vol. 61 (1955) p. 549.

Berkowitz, R. S. "Methods of Sampling Band-Limited Functions," Proc. IRE, Vol. 44 (1956), pp. 231–235.

Chakrabarty, N. B. "Synthesis of a Network for a Prescribed Time Function," *Indian J. Phys.*, Vol. 28 (1954), pp. 473–484 (2562–1955).

Chang, S. S. L. "Two Network Theorems for Analytical Determi nation of Optimum-Response Physically Realizable Networ Characteristics," Proc. IRE, Vol. 43 (1955), pp. 1128–1135.

Chu, Y. "Correlation between Frequency and Transient Response of Feedback Control Systems," *AIEE Trans. B Appl. Ind.* Vol. 72 (1953), pp. 81–92. (2427–1955).

Colombo, S. "Application de la Théorie des Distributions a quelque Problèmes de Physique Mathématique. La Mésure de Dirac e le Calcul des Phénomènes Transitoires," *Can. Phys.*, No. 60 (1955) pp. 23–40; No. 61, pp. 19–34.

Cotte, M. "Etude critique de la Notion de Vitesse de Groupe," *Rev. Gén. Sci.*, Vol. 62 (1955), pp. 237–245. (E78622).

Demczynski, S. "The Correlation between Decay-Time and Ampli tude Response," *Proc. IEE, C*, Vol. 103 (1956), pp. 64–70.

Heilfron, J. "On the Response of a Certain Class of Systems t Random Inputs", IRE Trans., Vol. IT-1 (1955), pp. 59–61.

Hellerman, H. "Transfer Functions for Amplitude–Modulate Signals," *AIEE Trans. Comm. Elect.* (1956), pp. 729–731.

Henze, E. "Einschwingvorgänge bei Amplitude- und Frequenz modulation an einigen speziellen passiven Netzuerken," *Arch Elektr. Ubertz.*, Vol. 9 (1955), pp. 326–338. (E77805).

Kahn, F. D. "The Correction of Observational Data for Instru mental Bandwidth," *Proc. Cambr. Phil. Soc.*, Vol. 51 (1955) pp. 519–525. (66, 1956).

Kaiser, J. J. "The Effect of Bandpass Limitation on Analogu Computer Solutions," *Bull. Amer. Math. Soc.*, Abstr. 783, Vo 61 (1955), p. 553.

Keiser, B. E. "The Linear, Input-Controlled, Variable-Pass Net work," IRE Trans., Vol. IT-1 (March, 1955), pp. 34–39. (D4297 1955).

— —, and Jeffrey, A. "The Linear, Input-Controlled Variable-Pas Network," IRE Trans., Vol. IT-2, (1956), pp. 42–44.

Klovski, D. D. "An Approximate Graphical-Analytical Metho for Plotting the Frequency Characteristic of a Linear Syste from its Transient Characteristic," *Zh. Tekh. Fiz.*, Vol. 25 (1955 pp. 333–338. (2558, 1955).

Kulikowsky, R. and Plebanski, J. "Optimum Characteristics o Linear Pulse Systems," *Bull. Acad. Pol. Sci.*, C1.IV, Vol. (1955), pp. 23–28. (E77860).

— — and Rybarski, A. "Criterion for Root Square Distortion a Limited Noise Power," *Archiwum Elektrotechniki 2*, tom. V (1956 pp. 379–400. (In Polish, English summary).

Lampard, D. G. "The Response of Linear Networks to Suddenl Applied Stationary Random Noise," IRE Trans., Vol. CT-(January, 1955), pp. 49–57.

Marique, J. "Forme des Signaux Radiotélégraphiques et Inter férence entres Voies Adjacentes," *Ann. Télécomm.*, Vol. 1 (1956), pp. 26–32.

rko, H. "Die Reziprozität von Zeit und Frequenz in der Nach-
chtentechnik," *Nachr. Tech. Z.*, Vol. 9 (1956), pp. 222–228,
66–271. (E85364, 86036).

vton, G. C. "Design of Control Systems for Minimum Band-
idth," *Trans. AIEE II*, Vol. 74 (1955), pp. 161–168. (D3994–
955).

enburger, R. (ed.). "Frequency Response," *Amer. Soc. Mech.
ng.*, (1956), 372 pp.

ers, J. "Einseitenband und Restseitenband Übertragung,"
Nachr. Tech. Fachber., Vol. 6 (1956), pp. 2–6.

senbloom, A., Heilfron, J., and Trautman, D. L. "Analysis of
Linear Systems with Randomly Varying Inputs and Prameters,"
955 IRE Convention Record, Part 4, pp. 106–113.

senbrock, H. H. "An Approximate Method for Obtaining
Transient Response from Frequency Response," *Proc. IEE*,
Vol. 102B (1955), pp. 744–751. (337, 1956).

ma, D. G. "Transient Response Calculation: Approximate
Method for Minimum-Phase-Shift Networks," *Wireless Eng.*, Vol.
2 (1955), pp. 179–186. (2559, 1955).

lettar, R. A. "Schrödinger-like Equation for the Measurement
Process," *Bull. Amer. Phys. Soc.*, Vol. 30 (1955), pp. 24–25.

omonoff, R. J. "Effect of Heisenberg's Principle on Channel
Capacity," Proc. IRE, Vol. 43 (April, 1955), p. 484. (2101–1955).

llard, D. V. "A Series Method of Calculating Control-System
Transient Response from the Frequency Response," *AIEE
Trans., Appl. Ind.*, No. 17 (1955), pp. 61–64.

gajewskaja, A. A. "Die Bestimmung der Amplituden-Phasen-
Characteristik linearer Systeme aus der Übergangsfunktion,"
Regl. t., Vol. 4 (1956), pp. 68–72.

ld, W. H. "Demonstration of Bandwidth Capabilities of Beyond-
Horizon Troposphere Radio Propagation," Proc. IRE, Vol.
43 (October, 1955), pp. 1297–1300.

eber, E. *Linear Transient Analysis*. New York: John Wiley &
Sons. Vol. I, 348 pp., 1954; Vol. 2, 452 pp., 1956.

vits, M. C. and Jackson, J. L., "Linear Filter Optimization with
Game Theory Considerations," 1955 IRE Convention Record,
Part 4, pp. 193–199.

ge, J. P. "Useful Bandwidth in Scatter Transmission," Proc. IRE,
Vol. 43 (November, 1956), pp. 1621–1622.

manian, A. H. "Further Bounds Existing on the Transient
Response of Various Types of Networks," Proc. IRE, Vol. 43
(March, 1955), pp. 322–327.

ennan, D. G. "On the Maximum Signal-to-Noise Ratios Realiz-
able from Several Noisy Signals," Proc. IRE, Vol. 43 (October,
1955), p. 1530.

nther, G. and Kraus, G. "Die Verbesserung des Signal/Geräusch
Verhaltnisses durch Wiederholung von Signalen," *Frequenz*,
Vol. 10 (1956), pp. 169–177.

ippov, L. J. "Le Minimum de Bruit dans la Réception Radio
electrique de Signaux d'Impulsions," *Radiotechnika*, Vol. 10
(1955), pp. 39–50.

c, M. "Signal and Noise Problems," *Proc. Symp. Special Topics
Appl. Math.*, Northwestern University, Chicago, Ill. 1953.
Amer. Math. Monthly, Vol. 61 (1954), pp. 23–26. (C53, 1955).

roni, B. "Considerazione sulla Efficienza di Modulazione,"
Conv. Elettro. Telev. Milan (1954), pp. 1177–1181.

zaris, S. D. *"Noise Figures from the View Point of Power Trans-
mission Efficiency,"* Quart Prog. Rep. M.I.T. Res. Lab. Elec-
tronics, (July, 1955), pp. 34–38.

ko, A. G. "Vergleich der Nachrichtenkapazität verschiedener
Modulations-systeme," *Radiotehnika*, Vol. 10 (1955), pp. 58–66.
(In Russian.) (E79348, D5314, 1955).

ax, N. "Signal-to-Noise Improvement and the Statistics of
Track Populations," *J. Appl. Phys.*, Vol. 26 (1955), pp. 586–595.
(2964–1955).

uko, A. G., see Sjuko, A. G.

rke, W. W. H. and Nikkel, R. F. "Mean Frequency Determi-
ation of Narrow-Band Noise Spectra," *Proc. IEE*, Vol. 102B
1955), pp. 364–370.

l, A. "Instantaneous Frequency and its Experimental Verifi-
ation," *Slaboproudy Obzor*, Vol. 16 (1955), pp. 345–350. (In
Czech.) (D4985, 1955).

inberg, H., Schultheis, P. M., Wogrin, C. A., and Zweig, F.

"Short-time Frequency Measurement of Narrow-Band Random
Signals of a Zero Counting Process," *J. Appl. Phys.*, Vol. 26
(1955), pp. 195–201. (2060, 1955).

d)

Daguet, J. "The Analytical Signal in Nonlinear Systems," *Proc.
Symp. Nonlinear Circuit Analysis* (1953), pp. 392–401.

Ville, J. A. "Sur le Prolongement des Signaux a Spectre Borne,"
Câbles et Transm., Vol. 10 (1956), pp. 44–52.

——, Bouzitat, J. "Sur un Type dâ Signaux Pratiquement Bornés
en Temps et en Frequence," *Câbles et Transm.*, Vol. 9 (1955),
pp. 293–303. (E78658).

III. DEFINITION, RELATION WITH STATISTICAL MECHANICS, PHILOSOPHY

Brillouin, L. "Information Theory and Negative Entropy," *Conv.
Elettron. Telev. Milano* (1954), pp. 1097–1132.

——. *Science and Information Theory*, New York: Academic Press,
320 pp., 1956.

Bakhmetev, M. M. "The Calculation of the Entropy of some
Special Probability Distributions," *Radiotehnika*, Vol. 1 (1956),
pp. 613–622.

Costa de Beauregard, O. "Sur l'Equivalence entre Information et
Entropie et sur l'Irreversibilité en Physique," *C. R. Acad. Sci.
Paris*, Vol. 243 (1956), pp. 1728–1730.

Faddeev, D. K. "On the Concept of Entropy of a Finite Probabilistic
Scheme Uspehi," *Mat. Nauk.* (*NS*), Vol. 11 (67) (1956), pp.
227–231. (In Russian.) (C1098, 1956).

Fast, J. D. and Stumpers, F. L. H. M. "Entropy in Science and
Technology," IV Entropy and Information, *Philips Tech. Rev.*,
Vol. 18 (1956/57), pp. 201–213.

MacCarthy, J. "Measures of the Value of Information," *Proc. Natl.
Acad. Sci.*, Vol. 42 (1956), pp. 654–655. (E87866).

MacKay, D. M. "Entropic Aspects of Information Theory,"
Nature, Vol. 178 (1956), p. 1422.

Mandelbrot, B. "On the Language of Taxonomy: An Outline of a
Thermostatistical Theory of Systems of Categories with Wiills
(Natural) Structure," *Proc. London Symp.* (1955), pp. 135–148.

——. "An outline of a Purely Phenomenological Theory of Statistical
Thermodynamics: Canonical Ensembles," IRE Trans., Vol.
IT-2 (September, 1956), pp. 190–204. Abstr. Proc. IRE, Vol.
44 (November, 1956), p. 1644.

Powers, K. H. "A Unified Definition of Information," *Quart.
Prog. Rep. M.I.T. Res. Lab. Electronics*, (July, 1955), pp. 42–48.

Price, R. "On Entropy Equivalence in the Time and Frequency
Domains," Proc. IRE, Vol. 43 (April, 1955), pp. 484–485.
(D3408, 1955).

Randow, Th. von. "The Mathematical Valuation of Information,"
Conv. Elettron. Telev. Milano (1954), pp. 1182–1186.

Sato, H. "Information Theory and Statistical Mechanics," *Rep.
Univ. Electro-Commun.*, Vol. 7 (1955), pp. 41–47. (In Japanese,
English summary.)

Seidler, J. "On the Concept of Entropy," *Archiv Elektrotechniki 2*,
tom V (1956), pp. 401–427. (In Polish, English summary.)

Tortrat, A. "Sur la Définition de l'Entropie en Théorie de l'Infor-
mation," *Ann. Télécomm.*, Vol. 10 (1955), pp. 39–47. (2740, 1955).

Touloukian, Y. S. *The Concept of Entropy in Communication,
Living Organisms and Thermodynamics*, Purdue University,
Eng. Bull. Res. Series No. 130, 1956.

IV. CORRELATION PREDICTION, FILTERING, STORAGE

Assejev, B. P. "Phasenkorrelationen, VEB Verlag 1957," *Abstr.
Nachr. Tech.*, Vol. 6 (1956), pp. 507–510.

Balakrishnan, A. V. and Drenick, R. F. "On Optimum Nonlinear
Extraction and Coding Filters," IRE Trans., Vol. IT-2 (Septem-
ber, 1956), pp. 166–177. Abstr. Proc. IRE, Vol. 44 (November,
1956), p. 1644.

Banerjee, R. B. "The Autocorrelation of Randomly Fading Waves,"
J. Atmos. Terr. Phys., Vol. 6 (1955), pp. 50–56. (1772, 1955).

Barber, N. F. "A Correlation Treatment of Fading Signals," *J.
Atmos. Terr. Phys.*, Vol. 8 (1956), pp. 318–330. (E86213).

Barrett, J. F. and Lampard, D. G. "On Expansion for Some Second-
Order Probability Distributions and its Application to Noise
Problems," IRE Trans., Vol. IT-1 (March, 1955), pp. 10–15.
Abstr. Proc. IRE, Vol. 43 (July, 1955), p. 892.

Bartlett, M. S. and Mehdi, J. "On the Efficiency of Procedures for Smoothing Periodograms from Time Series with Continuous Spectra," *Biometrika*, Vol. 42 (1955), pp. 143–150. (C1040, 1955).

——. "Problèmes de l'Analyse Spectrale des Series Temporelles Stationaires," *Publ. Inst. Stat. Univ. of Paris*, Vol. 3 (1954), pp. 119–134.

Berkowitz, R. S. "Methods of Sampling Band-Limited Functions," Proc. IRE, Vol. 44 (February, 1956), pp. 231–235.

Blum, M. "Generalization of the Class of Nonrandom Inputs of the Zadeh-Ragazzinni Prediction Model," IRE Trans., Vol. IT-2 (June, 1956), pp. 76–81.

——. "An Extension of the Minimum Mean Square Error Theory for Sampled Data," IRE Trans., Vol. IT-2 (September, 1956), pp. 176–184.

Bose, A. G. *Multiple Nonlinear Prediction*, Quart. Prog. Rep. M.I.T. Res. Lab. Electronics (January, 1956), pp. 77–80.

——, "A Theory for the Experimental Determination of Optimum Nonlinear Systems," *1956 IRE Convention Record*, Part 4, pp. 21–30.

——. *The Wiener Theory of Nonlinear Systems*, Quart. Prog. Rep. M.I.T. Res. Lab. Electronics, (July, 1955), pp 33–43 (October, 1955), pp. 55–62.

—— –. Jordan, K. L. *Measurements of Second-Order Correlation Functions*, Quart. Prog. Rep. M.I.T. Res. Lab. Electronics (April 1955), pp. 36–38.

Bracewell, R. N. "Correcting for Running Means by Successive Substitutions," *Austr. J. Phys.*, Vol. 8 (1955), pp. 329–334.

——. "Chord Construction for Correcting Aerial Smoothing," *Austr. J. Phys.*, Vol. 8 (1955), pp. 200–205.

Burr, E. J. "Sharpening of Observational Data in Two Dimensions," *Austr. J. Phys.*, Vol. 8 (1955), pp. 30–53.

Cerillo, M. V. "On the Synthesis of Linear Systems for Pure Transmission, Delayed Transmission and Linear Prediction of Signals," *Revista Mexicanna de Fisica*, Vol. 4 (1955) (In Spanish.) Quart. Prog. Rep. M.I.T. Res. Lab. Electronics, (October, 1956), pp. 66–89 (January, 1957).

Chałkovskii, V. I. "Reception of Pulse Signals by Means of the Mutual Correlation Method," *Radiotehnika*, Vol. 10 (1955), pp. 16–20. (In Russian) (D5358–1955) (E79381).

Chernov, L. A. "Correlation of Amplitude and Phase Fluctuations in Wave Propagation in a Statistically Inhomogeneous Medium," *C. R. Acad. Sci. U.R.S.S.*, Vol. 98 (1954), pp. 953–956. (989, 1955).

Cohen, J. W. "Some Calculations on a Method for Signal Testing. Inform Theory," *Proc. London Symp.* (1955), Abstr.

Daniels, F. B. "Existence of Periods in the Stock Market," *Science*, Vol. 121 (1955), pp. 151–154.

Davis, H. T. *The Analysis of Economic Time Series*, Evanston: Principia Press, Cowles Comm. Mon. No. 6, 620 pp., 1941.

Deutsch, R. "On a Method of Wiener for Noise through Nonlinear Devices," *1955 IRE Convention Record*, Part 4, pp. 186–192.

Drenick, R. F., Gartenhouse, S., and Nesbeda, R. "Detection of Coherent and Non-Coherent Signals," *1955 IRE Convention Record*, Part 4, pp. 182–185.

Fogel, L. J. "A Note on the Sampling Theorem," IRE Trans., Vol. IT-1 (March, 1955), pp. 47–48. Abstr. Proc. IRE, Vol. 43 (July, 1955), p. 893.

Franklin, G. "Linear Filtering of Sampled Data," *1955 IRE Convention Record*, Part 4, pp. 119–128.

George, S. F. "Effectiveness of Cross Correlation Detectors," *Natl. Electronics Conf.*, (1954), pp. 109–118. (E80216).

Goff, K. W. "An Analog Electronic Correlator for Acoustic Measurements," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 223–236. (2542–1955).

Gooder, A. W. "A Waveform Recorder Employing Sampling Techniques," *J. Brit. IRE*, Vol. 16 (1956), pp. 623–631.

Griffiths, J. W. R. "Optimum RC Filters for Separating Sinusoidal Signals from Noise," *Wireless Eng.*, Vol. 33 (1956), pp. 268–270.

Grosskopf, J., Kappelhoff, K. H., and Kopte, G. "Neue Geräte zur Ermittlung der Statistischen Verteilung Unregelmässiger Elektrischer Vorgänge," *Nachr. NTZ.*, Vol. 9 (1956), pp. 34–36.

Groves, G. W. "Numerical Filters for Discrimination Against Tidal Periodicities," *Trans. Amer. Geophys. Union*, Vol. 36 (1955), pp. 1073–1084.

Hayase, J. Y. *Properties of Second-Order Autocorrelation Functions.* Quart. Prog. Rep. M.I.T. Res. Lab. Electronics, (January, 1956), pp. 102–104.

——. *Field Mapping by Cross Correlation*, Quart. Prog. Rep. M.I.T. Res. Lab. Electronics, (July, 1955), pp. 52–58; (October, 1954), pp. 66–69.

——. *Calculation of Second-Order Autocorrelations Functions*, Quart. Prog. Rep. M.I.T. Res. Lab. Electronics, (January, 1955), pp. 45–49; (April, 1955), pp. 31–36.

——. *An Extension of the Wiener Theory of Nonlinear Filters*, Quart. Prog. Rep. M.I.T. Res. Lab. Electronics, (October, 1955), pp. 43–58.

——, Lorens, C. S. *Output Distributions of Linear Systems*, Quart. Prog. Rep. M.I.T. Res. Lab. Electronics, (October, 1956), pp. 53–58.

Herschel, R. "Uber ein verallgemeinertes Quadratisches Optimum," *Regl. t.*, Vol. 4 (1956), pp. 190–195.

Hontoy, P. and Lesuisse, W. "Théorie de l'Information, Caractéristiques Statistiques d'un Signal Sinussoidal Perturbé par le Bruit," *Rev. H. F. Brussels*, Vol. 3 (1955), pp. 27–54. (3402, 1955–E74550).

Horowitz, M. and Johnston, A. A. "Theory of Noise in a Correlation Detector," IRE Trans., Vol. IT-1 (September, 1955), pp. 3–5.

Horton, C. W. "Application of Correlation Techniques to Geophysical Prospecting," *J. Acous. Soc. Amer.*, Vol. 27 (1955), p. 198. (Abstr.).

Itô, H. "Observed values of Autocorrelation Function and Spectral Density," *J. Inst. Elect. Comm. Eng. (Japan)*, Vol. 36 (1953), pp. 667–672. (In Japanese) (D2049, 1955).

Johnson, K. R. "Optimum, Linear, Discrete. Filtering of Signals Containing a Nonrandom Component," IRE Trans., Vol. IT-2 (June, 1956), pp. 49–55. (E87862).

Karush, W. "Stability of a Method of Smoothing in a Digital Control Computer," IRE Trans., Vol. EC-4 (March, 1955), pp. 26–31. Abstr. Proc. IRE, Vol. 43 (April, 1955), p. 514.

Katz, L. "Determination of Visual Interpolation Errors in the Plotting of Curves from Commutated Data," IRE Trans., Vol. TRC-1 (February, 1955), pp. 15–24.

Kawata, T. "Remarks on Prediction Problems in the Theory of Stationary, Stochastic Processes," *Tôkohu Math. J.*, Vol. 6 (1954), pp. 13–20. (C151, 1955).

Kelner, R. C. and Glauberman, M. H. "Magnetic Shift-Register Correlator," *Electronics*, Vol. 29 (1956), pp. 172–175. (3286, 1956).

Klein, R. C. "Analog Simulation of Sampled Data Systems," IRE Trans., Vol. TRC-1 (May, 1955), pp. 2–7.

Korens, C. S. *Synthesis for Interference Filtering*, Quart. Prog. Rep. M.I.T. Res. Lab. Electronics (January, 1956), pp. 92–97.

Koschmann, A. H. "Time-Varying Filters for Nonstationary Signals on a Finite Interval," *1955 IRE Convention Record*, Part 4, pp. 102–105.

——, Truxal, J. G. "Optimum Linear Filtering of Nonstationary Time Series," *Natl. Electronics Conf.*, Vol. 10 (1954), pp. 119–127.

Kozulyaev, P. A. "On the Theory of Extrapolation of Stationary Sequences," *Moskov. Gos. Univ. Uč. Zap.*, Vol. 145 (1949), pp. 59–91, Mat 3. (In Russian) (C980, 1956).

Kramer, H. P. and Mathews, M. V. "A Linear Coding for Transmitting a Set of Correlated Signals," IRE Trans., Vol. IT-2 (September, 1956), pp. 41–46. Abstr. Proc. IRE, Vol. 44 (November, 1956), p. 1643.

Kulikowski, R. "Minimum Signal Distortion and Noise Power in Linear Filters," *Bull. Acad. Sci. Pol.*, Cl IV, (1956), pp. 123–126. (E87011).

Lago, G. V. "Additions to Sampled Data Theory," *Natl. Electronics Conf.*, Vol. 10 (1954), pp. 758–766.

Laning, J. H. J. and Battin, R. H. "An Application of Analog Computers to the Statistical Analysis of Time Variable Networks," IRE Trans., Vol. CT-2 (March, 1955), pp. 44–49. Abstr Proc. IRE, Vol. 43 (May, 1955), p. 641.

Lampard, D. G. "A New Method of Determining Correlation Functions of Stationary Time Series," *Proc. IEE*, C102 (1955), pp. 35–41. (2352, 1955).

——. "The Probability Distribution for the Filtered Output of a Multiplier Whose Inputs are Correlated Stationary, Gaussian Time Series," IRE Trans., Vol. IT-2 (March, 1956), pp. 4–11.

Lange, F. H. "Uber die Anwendung der Korrelationsanalyse in der Nachrichtentechnik," *Nachrichtentechnik*, Vol. 6 (1955), pp. 445–449. (E79299); Vol. 6 (1956), pp. 8–13. *Korrelationsanalysatoren, Nachr. t.*, Vol. 6 (1956), pp. 148–155. *Korrelationsanalyse und Vierpoltheorie, Nachr. t.*, Vol. 6 (1956), pp. 315–319. (E86908).

Lee, Y. W., Bose, A. G., and Hayase, J. Y. *Properties of Secon-*

rder Autocorrelation Functions. Quart. Prog. Rep. M.I.T. Res.
Lab. Electronics (October, 1954), pp. 63–66.

—. *Autocorrelation of Poisson Waves,* Quart. Prog. Rep. M.I.T.
Res. Lab. Electronics, (January, 1955), pp. 56–62.

s, A. B. "Interpolation and Extrapolation of Sampled Data,"
IRE Trans., Vol. IT-2 (March, 1956), pp. 12–17.

nov, Y. P. "On Filtering Nonstationary Random Functions,"
utomat. i. Telemeh., Vol. 14 (1956), pp. 97–106. (In Russian.)
C1100, 1956).

yd, S. P. and Mac Millan, B. "Linear Least Squares Filtering
nd Prediction of Sampled Signals," *Proc. Symp. P.I.B.,* Vol.
(April, 1955), pp. 221–247.

c Fadden, J. A. "The Correlation Function of a Sine Wave plus
Noise after Extreme Clipping," IRE Trans., Vol. IT-2 (June,
956), pp. 82–83. (E87859).

ck, C. L. "Diversity Reception in UHF Long-Range Communi-
ation," Proc. IRE, Vol. 43 (October, 1955), pp. 1281–1289.

lone, T. F. "Application of Statistical Methods in Weather
Prediction," *Proc. Natl. Acad. Sci.,* Vol. 41 (1955), pp. 806–815.

nn, H. B. "A Theory of Estimation for the Fundamental
Random Process and the Ornstein-Uhlenbeck Process," *Sankya,*
Vol. 13 (1954), pp. 324–350.

rko, H. "Korrelation und Vorausbestimmung von Signalen,"
Nachr. Tech. Fachber, Vol. 6 (1956), pp. 7–12.

Ginnis, C. E. *Correlation Detection of a Periodic Signal Multiplied
by Noise,* Quart. Prog. Rep. M.I.T. Res. Lab. Electronics
(October, 1954), pp. 69–73.

llen, G. L., Morrow, W. E., Poté, A. J., Radford, W. H., and
Wiesner, J. B. "UHF Long-Range Communication Systems,"
Proc. IRE, Vol. 43 (October, 1955), pp. 1269–1281.

yer-Eppler, W. and Darius, G. "Die Autokorrelation von ebenen
zweidimensionalen Bildvorlagen," *Nachr. tech. Fachber,* Vol. III
(1956), pp. 40–47.

—. "Two-dimensional Photographic Autocorrelation of Pictures
and Alphabet Letters," *Proc. London Symp.* (1955), pp. 34–36.

ller, K. S. and Zadeh, L. A. "Solution of an Integral Equation
occurring in the Theories of Prediction and Detection," IRE
Trans., Vol. IT-2 (June, 1956), pp. 72–75. (E87760).

amias, V. "Application des Fonctions Impulsives à la Théorie du
Filtrage et de la Prevision," *Ann. Télécomm.,* Vol. 10 (1955),
pp. 135–148. (E77093).

wers, K. H. *An Application of Prediction Theory to Information
Rates,* Quart. Prog. Rep. M.I.T. Res. Lab. Electronics, (January,
1956), pp. 80–91.

ice, R. "A Note on the Envelope and Phase—Modulated Com-
ponents of Narrow-Band Gaussian Noise," IRE Trans., Vol.
IT-1 (September, 1955), pp. 9–13.

verson, J. L. "Linear Complementary Smoothing Compensated
for Sampled Data Lags," 1956 IRE Convention Record,
Part 2, pp. 106–108.

hmelovsky, K. H. "Einfluss der Korrelation swischen den Emp-
fangsfeldstärken bei Diversity-Empfang," *Hoch. u Elektroak,*
Vol. 65 (1956), pp. 74–76.

thuraman, R. "Correlation Functions and their Uses," *J. Inst.
Telecomm. Eng. (New Delhi),* Vol. 2 (1956), pp. 87–92. (D4580,
1956).

ith, W. B. *Measurement of Second-Order Autocorrelation Func-
tions,* Quart. Prog. Rep. M.I.T. Res. Lab. Electronics (January,
1955), pp. 49–55.

aras, H. "Diversity Reception with Correlated Signals," *J. Appl.
Phys.,* Vol. 27 (1956), pp. 93–94.

ein, S. and Storer, J. E. "Generating a Gaussian Sample," IRE
Trans., Vol. IT-2 (1956), pp. 87–90. (E87861).

evens, S. S. "On the Averaging of Data," *Science,* Vol. 121 (1955),
pp. 113–116.

ewart, R. M. "Statistical Design and Evaluation of Filters for
the Restoration of Sampled Data," Proc. IRE, Vol. 44 (February,
1956), pp. 253–257.

kano, K. "Note on Wiener's Prediction Theory," *Ann. Inst.
Statist. Math. Tokyo,* Vol. 5 (1954), pp. 67–72. (C151, 1955).

cker, D. G. and Griffiths, J. W. R. "Signal/Noise Performance
of Multiplier (or Correlation) and Addition (or Integrating)
Types of Detector," *Proc. IEE* C 102 (1955), pp. 181–190.
(E78673).

e, L. A. "Weighted Least—Squares Smoothing Filters," IRE
Trans., Vol. CT-2 (June, 1955), pp. 197–203.

baniak, K. and Zieba, A. "Prediction of Solar Activity," *Arch.*

Elektrotech., Vol. 2 tom V (1956), pp. 356–364. (In Polish, English
summary.)

Wernikoff, R. E. *Effects of Periodic Sampling on Output Noise in
Autocorrelation Detection,* Quart. Prog. Rep. M.I.T. Res. Lab.
Electronics (April, 1954), pp. 51–53; (July, 1954), pp. 55–59.

Whittle, P. "The Analysis of Multiple Stationary Time Series,"
J. Roy. Statist. Soc. B, Vol. 15 (1953), pp. 125–139.

Wiener, N. "On the Factorization of Matrices," *Comm. Math.
Helv.,* Vol. 29 (1955), pp. 97–111.

Yaglom, A. M. "Theory of Extrapolation and Filtering of Random
Processes," *Ukrain. Math. Z.,* Vol. 6 (1954), pp. 43–57. (In
Russian).

—. "Extrapolation, Interpolation and Filtering of Stationary
Random Processes with Rational Spectral Density," *Trudy.
Moskov. Mat. Obšč.,* Vol. 4 (1955), pp. 333–374. (In Russian).

Yerg, D. G. "Notes on Correlation Methods for Evaluating Iono-
spheric Winds from Radio Fading Records," *J. Geophys. Res.,*
Vol. 60 (1955), pp. 173–185. (2955-1955).

Zemanian, A. H. "A Note on the Bounds of Autocorrelation Func-
tions," IRE Trans., Vol. IT-1 (June, 1955), pp. 48–49.

V. a) RADAR;   b) RADIONAVAGATION

a)

Benner, A. H. and Drenick, R. F. "On the Problem of Optimum
Detection of Pulsed Signals in Noise," *RCA Rev.,* Vol. 16 (1955),
pp. 461–479. (439, 1956) (E79359).

Borg, H. "Sur les Possibilités Techniques de Réduction du Bruit
dans la Réception des Signaux Faibles de Radar," *Ann. Télécomm.,*
Vol. 11 (1956), pp. 90–110.

Bussgang, J. J. and Middleton, D. "Optimum Sequential Detection
of Signals in Noise," IRE Trans., Vol. IT-1 (September, 1955),
pp. 5–18.

Bystrov, G. N. "Connection between the Detectability of an Object
and the Number of Illuminating Pulses," *Radiotehnika Moscov.,*
Vol. 11 (1956), pp. 74–76. (3067, 1956).

Cox, C. P. "Audar," *Radio. Telev. News,* Vol. 54 (March, 1955),
pp. 31–33. (E77313).

Delano, R. H. and Pfeffer, I. "The Effect of A.G.C. on Radar
Tracking Noise," 1955 IRE Convention Record, Part 4,
pp. 200–208.

Dinneen, G. P. and Reed, I. S. "An Analysis of Signal Detection
and Location by Digital Methods," IRE Trans., Vol. IT-2
(March, 1956), pp. 29–38.

Duwaldt, B. J. "Inverse Probability in Angular-Tracking Radars,"
IRE Trans., Vol. IT-2 (March, 1956), pp. 38–42.

Feiner, A. and Diamond, F. "A Three-Dimensional Aircraft Visi-
bility Diagram," 1956 IRE Convention Record, Part 8, pp.
97–100.

Freeman, J. J. "On Power Spectra and the Minimum Detectable
Signal in Measurement Systems," *J. Appl. Phys.,* Vol. 26 (1955),
pp. 236–240. (2053, 1955).

Hall, W. M. "Prediction of Pulse Radar Performance," Proc. IRE,
Vol. 44 (February, 1956), pp. 224–230.

Harmuth, H. F. *Integration Method of Detection and its Application
to the Transmission of Teletype Signals,* Signal Corps 101 pp.,
May, 1955. (PB 118617).

—. "A New Teletypewriter using the Integration Method of
Detection," 1955 IRE Convention Record, Part 8, pp. 52–56.
(E77913).

—. "Detection of Pulse code Modulation by the Integration
Method," *Natl. Electronics Conf.,* Vol. 10 (1954), pp. 268–292.

Harrington, J. V. "An Analysis of the Detection of Repeated
Signals by Binary Integration," IRE Trans., Vol. IT-1 (March,
1955), pp. 1–9.

Heaps, H. S. "The Effect of a Random Noise Background upon the
Detection of a Sinusoidal Signal," *Can. J. Phys.,* Vol. 33
(1955), pp. 509–520. (D4906-1955).

Helstrom, C. W. "The Resolution of Signals in White Gaussian
Noise," Proc. IRE, Vol. 43 (1955), pp. 1111–1118. (3737, 1955).

Herold, H. "Rauschabstandsvergrösserung bei Radar Geräten,"
Nachrichtentech. Fachber, Vol. 2 (1955), pp. 19–23. (E88205).

Jaffe, R. and Rechtin, E. "Design and Performance of Phase-Lock
Circuits Capable of Near-Optimum Performance over a Wide-
Range of Input-Signal and Noise Level," IRE Trans., Vol.
IT-1 (March, 1955), pp. 66–76.

James, R. H. "Audar," *Industr. Ital. Electrotec.*, Vol. 8 (1955), pp. 15–19. (E78407).

Kaplan, E. L. "Signal-detection Studies with Applications," *Bell Sys. Tech. J.*, Vol. 34 (1955), pp. 403–437. (D3406, 1955), (2631, 1955).

Katzin, M. "Recent Developments in the Theory of Sea Clutter," 1956 IRE Convention Record, Part 1, pp. 19–20.

Levine, D. "The Radar Display as a Linear Filter," IRE Trans., Vol. ANE-3 (September, 1956), pp. 124–127. Abstr. Proc. IRE, Vol. 44, (1956), p. 1897.

Loos, R. H. "Die Berechnung Linearer Schaltungen zum Ausfiltern von Signalen aus einem Rauschpegel," *Staboproudy Obzor*, Vol. 16 (1955), pp. 22–33. (In Czech).

MacDonald, F. C. "The Correlation of Radar Sea Clutter on Vertical and Horizontal Polarization with Wave Height and Slope," 1956 IRE Convention Record, Part 1, pp. 29–32.

Marino, A., Steinel, K., Meinke, H. H., and Schröter, F. "Radar und Nachrichtentechnik," *Radio Mentor*, Vol. 21 (1955), pp. 382–386. (E77022).

——. "Sending Frequencies and Channel Width in Radar Techniques," *La Ricerca Scientifica*, Vol. 25 (1955), pp. 1959–1976.

Matthews, W. E. "Analytical Prediction of Missile Guidance Accuracy," 1956 IRE Convention Record, Part 8, pp. 150–153.

Maximon, L. C. and Ruina, J. P. "Some Statistical Properties of a Signal Plus Narrow-Band Noise Integrated over a Finite Time Interval," *J. Appl. Phys.*, Vol. 27 (1956), pp. 1442–1448.

Mechler, E. A. "Information rate in Remoted Radar Systems," IRE Trans, Vol. CS-4 (December, 1956), pp. 120–128. (E87278).

Meinke, H. "Ubertragung von Radar-Schirmbildern mit komprimierter Bandbreite," *Nachr. Tech. Z.*, Vol. 9 (1956), pp. 214–221. (E85558).

Middleton, D. and van Meter, D. "Detection and Extraction of Signals in Noise from the Point of View of Statistical Decision Theory," *J. Soc. Ind. Appl. Math.*, Vol. 3 (1956), pp. 192–253; Vol. 4 (1956), pp. 86–119.

——, ——. "On Optimum Multiple-Alternative Detection of Signals in Noise," IRE Trans., Vol. IT-1 (September, 1955), pp. 1–8.

Miller, K. S. and Bernstein, R. "Coherent Detection of Sinusoidal Signals in Gaussian Noise," 1956 IRE Convention Record, Part 4, pp. 12–14.

Phillips, M. L. "Estimating the Ratio of Steady Sinusoidal Signal to Random Noise from Experimental Data," Proc. IRE, Vol. 44 (June, 1956), p. 692.

Rooney, T. J., Rider, L. S. and Rudwick, B. H. "Cumulative Probability of Radar Detection," *Tele-Tech.*, Vol. 14 (April, 1955), pp. 70–72, 184. (2963–1955).

Root, W. L. and Pitcher, T. S. "Some Remarks on Statistical Detection," IRE Trans., Vol. IT-1 (December, 1955), pp. 33–38.

Schooley, A. H. "Some Limiting Cases of Radar Sea Clutter Noise," Proc. IRE, Vol. 44 (August, 1956), pp. 1043–1047. (E87281) (3390, 1956).

Schwartz, M. "Effects of Signal Fluctuation on the Detection of Pulsed Signals in Noise," IRE Trans., Vol. IT-2 (June, 1956), pp. 66–71. (E87864).

Sherman, H. "Some Optimal Signals for Time Measurement," IRE Trans., Vol. IT-2 (March, 1956), pp. 24–28.

——. *The Detection and Phase Determination of Signals in Additive and Multiplicative Noise*, Thesis Doctor Elec. Eng., Polytech. Inst. Brooklyn, 1955.

Sherwin, C. W., Kodman, F., Jr., Kovaly, J. J., Prothe, W. C., and Melrose, J. "Detection of Signals in Noise: A Comparison between the Human Detector and an Electronic Detector," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 617–622. (A7237, 1956).

Siebert, W. C. "A Radar Detection Philosophy," IRE Trans., Vol. IT-2 (September, 1956), pp. 204–221, Abstr. Proc. IRE, Vol. 44 (November, 1956), p. 1644.

Stone, W. M. "On the Probability of Detection with a Postdetection Filter," *Bull. Amer. Math. Soc.* Abstr. 621, 61, 441, 1955.

Swets, J. A. and Birdsall, T. G. "The Human Use of Information III Decision-Making in Signal Detection and Recognition Situations Involving Multiple Alternatives," IRE Trans., Vol. IT-2 (September, 1956), pp. 138–165. Abstr. Proc. IRE, Vol. 44 (November, 1956), p. 1644.

Tucker, D. G. and Griffiths, J. W. R. "Detection of Pulse Signals in Noise Theory of Intensity-Modulated Display in Terms of Just-noticeable Differences," *Wireless Eng.*, Vol. 32 (1955), pp. 290–297. (E79358) (129, 1956).

b)

Bolie, V. W. "A Radiometric Inertial Reference System," Proc. IRE, 1956 IRE Convention Record, Part 8, pp. 139–149.

Rosenberg, S. "Considerations Affecting the Choice of a Long Range Navigation System," 1956 IRE Convention Record, Part 8, pp. 154–164.

VI. a) SPEECH,   b) HEARING,   c) VISION,   d) LINGUISTICS, SEMANTICS

a)

Allen, V. H. "Laboratory Equipment for Quantizing Speech," *Electronic Eng.*, Vol. 28 (1956), pp. 48–52.

Ayers, W. "New Ideas in the Transmission of Speech," *Post. Off. Telecommunic J.*, Vol. 8 (1956), pp. 171–174. (E88521).

Black, J. W. and Moore, W. E. *Speech: Code, Meaning and Communication*, New York: McGraw-Hill, 430 pp., 1955.

——. "The Reception of Repeated and Overlapping Speech Patterns," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 494–496 (2818, 1955).

——. "The Information of Sounds and Phonetic Diagrams of One- and Two-Syllable Words," *J. Speech Hearing Disorders*, Vol. 19 (1954), pp. 397–411. (B29, 1955, 5615).

Bogart, B. P. "The Vobanc—a two to one Speech Bandwidth Reduction System," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 399–404. (E87015).

Booth, A. D. "Recognizing Spoken Sounds by means of a Computer," *Comp. and Autom.*, Vol. 4 (1955), pp. 9, 31.

Borst, J. M. "The Use of Spectrograms for Speech Analysis and Synthesis," *J. Audio Eng.*, Vol. 4 (1956), pp. 14–23. (E88520).

Broadbent, D. E., Ladefoged P., and Lawrence, W. "Vowel Sounds and Perceptual Constancy," *Nature*, Vol. 178 (1956), pp. 815–816. (E88517).

Challier, L., "Le 'Typosonographe' Phonétique au Phonétographe," (Le Congrès International de l'Automatique de Paris 1956), *Automatisme, Fr.*, (1956), pp. 302–305. (E86850).

Chao, Y. R. "Linguistic Prerequisites for a Speech writer," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 1107–1109.

Chang, S. H. "Two Schemes of Speech Compression Systems," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 565–572. (A6476, 1956).

Cherry, E. C. and Marland, P. M. "Experiments on the Complete Suppression of Stammering," *Nature*, Vol. 176 (1955), p. 874. (E79813).

David, E. E., Jr. and McDonald, H. S. "A Bit-Squeezing Technique applied to Speech Signals," 1956 IRE Convention Record, Part 4, pp. 148–153.

Deman, P. "Modulation à Amplitude Constante en Téléphonie," *Onde Elec.*, Vol. 35 (1955), pp. 739–746. (E77850).

Dolansky, L. O. "Electronically Controlled Audio-Filter," Proc. IRE, Vol. 43 (November, 1955), pp. 1580–1586. 1955 IRE Convention Record, Part 7, pp. 41–48. (372, 1956).

Dudley, H. "Fundamentals of Speech Synthesis," *J. Audio Eng. Soc.*, Vol. 3 (1955), pp. 170–185. (E85752).

Dukes, J. M. C. "The Effect of Severe Amplitude Limitation on Certain Types of Random Signal: a Clue to the Intelligibility of Infinitely Clipped Speech," *Proc. IEE*, C 102, (1955), pp. 88–97. (2412, 1955).

Endres, W. "Ueber die Grösse des 'Weitschweifigen' Anteils in der Sprache und seine Bedeuting für den Fernsprechweitverkehr," *Fernmeldetechn. Z.*, Vol. 8 (1955), pp. 89–93. (D3865, 1955).

Essler, W. O. "A Versatile Audio Spectrometer," IRE Trans., Vol. AU 3 (March-April, 1955), pp. 24–28. Abstr. Proc. IRE, Vol. 43 (April, 1955), p. 513.

Ewertsen, H. W. "Delayed Speech Test," *Acta Oto-Laryngologica*, Vol. 45 (1955), pp. 383–387.

Flanagan, J. L. "Automatic Extraction of Formant Frequencies from Continuous Speech," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 110–118.

——. "Evaluation of Two Formant-Extracting Devices," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 118–125.

——. *A Speech Analyser for a Formant-Coding Compression System*, Sc. D. Thesis, Sci. Rep. 4, Acous. Lab. M.I.T.

——. "A Difference Limen for Vowel Formant Frequency," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 613–677.

——. "Bandwidth and Channel Capacity Necessary to Transmit the Formant Information of Speech," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 592–596. (A6483, 1956).

——, House, A. S. "Development and Testing of a Formant-Coding Speech Compression System," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 1099–1106.

ry, D. B. and Denes, P. "Experiments in Mechanical Speech Recognition," *Proc. London Symp.*, (1955), pp. 206–214.

alle, M. and Hughes, G. W. *Vowel Identifier*, Quart. Prog. Rep. M.I.T. Res. Lab. Electronics (October, 1956), pp. 109–117.

——, ——. "Spectral Properties of Fricative Consonants," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 303–310. (A3587, 1956) (E85756).

oward, C. R. "Speech Analysis Synthesis Schemes Using Continuous Parameters," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 1091–1098.

ock, W. E. "Speech Bandwidth Compression," *Bell Labs. Rec.*, Vol. 34 (1956), pp. 81–85.

upfmüller, K, Warns. "Sprachsynthese aus Lauten," *Nachr. Techn. Fachber.*, Vol. III (1956), pp. 28–32.

ange, F. H. "Neue Untersuchungsmethoden in der Elektroakustik," *Nachr. Technik*, Vol. 6 (1956), pp. 388–392.

aeda, K. "Relation between Articulation and Intelligibility and Entropy of Speech Words," *J. Inst. Elec. Comm. Eng. Japan*, Vol. 37 (1954), pp. 359–365 (In Japanese.) (D2052, 1955).

arcou, P. and Daguet, J. "New Methods of Speech Transmission," *Proc. London Symp.* (1955), pp. 231–244.

——, ——. "Une Nouvelle Méthode de Transmission de la Parole," *Ann. Télécomm.*, Vol. 11 (1956), pp. 118–126.

eyer-Eppler, W. and Luchsinger, R. "Beobachtung bei der verzögerten Ruckkopplung der Sprache," *Folia Phoniatrice*, Vol. 7 (1955), pp. 87–99.

iller, G. A. "Information Theory and the Study of Speech," *Current Trends*, pp. 119–139. (B29, 1955, 835).

ine, H. "Representation of a Noisy Discrete Channel and its Application to Articulation," *J. Inst. Elect. Comm. Eng. Japan*, Vol. 37 (1954), pp. 353–358 (In Japanese.) (D2051, 1955).

ol. H. "Speech and Hearing, a Unity," *P.T.T. Bedrijf*, Vol. 7 (1956), pp. 69–75. (D4581, 1956).

oser, H. M. and Dreher, J. J. "Phonemic Confusion Vectors," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 874–881. (E79810).

olson, H. F. and Belar, H. "Phonetic Typewriter," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 1072–1081.

eterson, E. G. "Systematic Research in Experimental Phonetics: 4. The Evaluation of Speech Signals," *J. Speech, Hearing Disorders*, Vol. 19 (1954), pp. 158–168. (B29, 1955, 3552).

anke, O. F. and Lulhas, H. "*Gehör, Stimme, Sprache*," Springer, 302 p., 1953 (E-L3768).

ichards, D. L. "A Development of the Collard Principle of Artificial Articulation," *Proc. IEE* B 103, (1956), pp. 679–691. (E88518).

chneider, H. "Die Verständlichkeit Amplitudenbegrenzter Sprache," *Frequenz*, Vol. 10 (1956), pp. 97–106, 152–161.

tevens, K. N. and House, H. S. "Development of a Quantitative Description of Vowel Articulation," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 484–493. (2559, 1955).

——, Bastide, R. P., and Smith, C. P. "Electrical Synthesizer of Continuous Speech," *J. Acous. Soc. Amer.*, Vol. 27 (1955), p. 207. (Abstr.).

——. "Synthesis of Speech by Electrical Analog Devices," *J. Audio Eng. Soc.*, Vol. 4 (1956), pp. 2–8. (E88519).

ebra, W. "Spraakanalyse en Spraaksynthese," *Polytech. Tij d-schrift*, Vol. A 10 (1955), pp. 292a–299a.

omatis, A. "Relation entre l'Audition et la Phonation," *Ann. Télécomm.*, Vol. 11 (1956), pp. 151–158. (E85754).

ilbig, F. "An Analysis of Clipped Speech," *J. Acous. Soc. Amer.*, Vol. 27 (1955), p. 207. (Abstr.).

——, Haase, K. H. "Ueber einige Systeme zur Sprachbandkompression," *Nachr. tech. Fachber.*, Vol. III (1956), pp. 81–93.

——, ——. Some Systems for Speech Band Compression. *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 573–577. (A6477, 1956).

iren, J. and Stubbs, H. L. "Electronic Binary Selection-System for Phonetic Classification," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 1082–1091.

amoto, J. "Statistical Characteristics of Telephone Speech," *Rep. Elec. Comm. Lab. Japan*, Vol. 3 (September, 1955), pp. 15–18.

)

llanson, J. T. and Whitfield, I. C. "The Cochlear Nucleus and its Relation to Theories of Hearing," *Proc. London Symp.* (1955), pp. 269–286.

Anon, "Research on Hearing the Loudness of Sound," *Instrument Practise*, Vol. 10 (1956), p. 135.

Axon, P. E., Gilford, C. L. S. and Shorter, D. E. L. "Artificial Reverberation," *Proc. IEE* B 102 (1955), pp. 624–642. (E79053) (3476, 1955).

Avakian, R. V. "Measurement of Threshold Sound Intensities and Differential Frequency Thresholds by means of Momentary Conditioned Reflexes in a Human," *Probl. Physiol. Acoustics*, U.S.S.R., Vol. III, 1956.

Bachem, A. "Absolute Pitch," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 1180–1185. (A1006, 1955).

——. "Time Factors in Relative and Absolute Pitch Determination," *J. Acous. Soc. Amer.*, Vol. 26 (1954), pp. 751–753.

Bekesy, G., von "Current Status of Theories of Hearing," *Science*, Vol. 123 (1956), pp. 779–783. (A7913, 1956).

Broadbent, D. E. "Growing Points in Multichannel Communication," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 533–535.

Bruce, D. J. "The Effects of Context upon the Intelligibility of Heard Speech," *Proc. London Symp.* (1955), pp. 245–252.

Cherry, E. C. and Sayers, B. M. A. " 'Human Cross-Correlator,' a Technique for Measuring Certain Parameters of Speech," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 889–895.

Chocholle, R. "Etude Statistique de la Sensibilité Auditive Différentielle d'Intensité," *Acustica*, Vol. 5 (1955), pp. 134–141. (E77503).

Christman, R. J. "The Perception of Direction as a Function of Binaural Temporal and Amplitude Disparity," 1956 IRE Convention Record, Part 9, pp. 3–7.

Dadson, R. S. "The Normal Threshold of Hearing and other Aspects of Standardization in Audiometry," *Acustica*, Vol. 4 (1954), pp. 151–154.

David, E. E. and McDonald, H. S. "A Bit-Squeezing Technique Applied to Speech Signals," 1956 IRE Convention Record, Part 4, pp. 148–153.

Duyff, J. W., Gemert, A. G. M., van, and Schmidt, P. H. "Binaural Sound Location in the Horizontal Plane," *Acta Physiol. Pharmacol. Neerl.*, Vol. 1 (1950), pp. 540–561.

Egan, J. P., Clarke, F. R., and Carterette, E. C. "On the Transmission and Confirmation of Messages in Noise," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 536–550. (A6470, 1956).

Enkel, F. "Ueber den Informationsgehalt der Geräuschlaute," *Nachr. Tech. Z.*, Vol. 9 (1956), pp. 493–498.

——. "Ein Beitrag zur Typologie des Gehörs," *Nachr. Tech. Fachber.*, Vol. III (1956), pp. 3–7.

Fack, H. "Informationstheoretische Behandlung des Gehörs," *Imp. t.*, pp. 289–338. See Winckel p. 46 (XI).

Fairbanks, G. House A. S., and Melrose, J. "Auditory Detection of the Presence and Absence of Signals in Noise," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 614–616. (A6487, 1956).

Feldtkeller, R. and Zwicker, E. "Das Ohr als Nachrichtenempfänger," *Monographien der Elektr. Nachr. Technik Bd.*, Vol. 19, 112 S, S. Hirzel (Stuttgart), 1956.

——. "Ueber die Zerlegung des Schallspektrums in Frequenzgruppen durch das Gehör," *Eletkr. Rdschau*, Vol. 9 (1955), pp. 387–389.

Frishkopf, L. S., Rosenblith, W. A., and Brown, K. H. *Electrophysiological Studies of the Auditory Nervous System*, Quart. Prog. Rep. M.I.T. Res. Lab. Electronics (October, 1953), pp. 45–49.

——, ——. *A Probabilistic Approach to Certain Neuroelectric Phenomena*, Quart. Prog. Rep. M.I.T. Res. Lab. Electronics (October, 1954), pp. 87–94.

Gemert, A. G. M., van. "Absolute and Differential Thresholds for White Noise," *Acta Physiol. Pharmacol. Neerl.*, Vol. 4 (1955), pp. 274–290.

Gershuni, G. V. "Some New Methods of Measuring Human Hearing and Results Obtained from Them," *Akust. Zh.*, Vol. 2 (1956), pp. 137–141. (In Russian). (D8539, 1956).

——. "Study of the Action of the Human Auditory Analyzer on the Basis of Using Various Reactions," *Probl. Physiol. Acous.* U.S.S.R., Vol. III, 1956.

Goff, K. W. "The Application of Correlation Techniques to some Acoustical Measurements," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 236–246.

Huizinga, E. "The Sense of Hearing—Its Significance in Human

Beings," *Proc. Int. Course Paedo-Audiology Groningen Univ.*, (1953), pp. 9–15. (B29, 322, 1955).

Jerger, J. F. "Differential Intensity Sensitivity in the Ear with Loudness Recruitment," *J. Speech Hearing Disorders*, Vol. 20 (1955), pp. 183–191. (E77502).

Kallenbach, W. "Schallübertragung und Informationstheorie," *Phys. Blätter*, Vol. 10 (1954), pp. 486–493.

Kiang, N. Y. S. and Goldstein, M. H. *Responses from the Auditory Nervous System to Bursts of Noise.* Quart. Prog. Rep. M.I.T. Res. Lab. Electronics (January, 1956), pp. 138–141.

Klump, R. G. and Eady, H. R. "Some Measurements of Interaural Time Difference Thresholds," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 859–860.

Koefoed, J. "A 'Neuronic Model' of the Inner Ear," *Proc. London Symp.*, (1955), p. 401.

Licklider, J. C. R. "Auditory Frequency Analysis," *Proc. London Symp.*, (1955), pp. 253–268.

Masureva, A. J. "The Problem of Measuring Discrimination Limits for Auditory Signals in a Human," *Probl. Physiol. Acous. U.S.S.R.*, Vol. III, 1955.

Meyer, M. F. "Neurological Theory of Beat Tones," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 877–882.

Meyer-Eppler, W. "Experimentelle Untersuchungen zum Mechanismus von Stimme und Gehör in der lautsprachlichen Kommunikation," *Forschungsber. des Wiss. und Verkehrsmin.* Nr 221–199.

——. "Statistische und Psychologische Klangprobleme," *Die Reihe* (Univ. Edit., Wien) Heft 1, Elektronische Musik (1955), pp. 22–28.

——. (ed.). Gravesano: *Musik-Raumgestaltung-Elektroakustik*, (20 Vorträge, 141 pp.). Ars-Viva Verlag (Scherchen) Mainz.

Michels, W. C. "Rating Scale Method for Comparative Loudness Measurements, *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 1173–1180.

Miller, G. A. and Nicely, P. E. "An Analysis of Perceptual Confusion among English Consonants," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 338–352.

Miller, I. "Auditory Perception in Relation to Amount of Information and Speech to Noise Ratio," *Diss. Abstr.*, Vol. 14 (1954), p. 2316, Ph.D. Thesis, Purdue University, 1954.

Mol, H. "The Haircells and Intelligibility," *Acta Oto-Laryngologica*, Vol. 46 (1956), pp. 416–421.

Moles, A. "La Structure Physique du Signal Musical," *Rev. Sci.*, Vol. 91 (1953), pp. 277–303.

Moser, H. N. and Dreher, J. J. "Effects of Training on Listeners in Intelligibility Studies," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 1213–1219.

Mowbray, G. H. Gebhardt, J. W., and Byham, C. L. "Sensitivity to Changes in the Interruption Rate of White Noise," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 106–110.

Olson, H. F. and Belar, H. "Electronic Music Synthesizer," *J. Acous. Soc. Amer.*, Vol. 27 (1956), pp. 595–612.

Palmer, J. M. "The Effect of Speaker Differences on the Intelligibility of Phonetically Balanced Word Lists," *J. Speech Hearing Disorders*, Vol. 20 (1955), pp. 192–195. (E77506).

Penn, J. P. "Voice and Speech Patterns of the Hard of Hearing," *Acta Oto-Laryngologica Suppl.*, Vol. 124 (1955), p. 69.

Pikler, A. G. "Information Analysis and Power-Spectrum Model of Hearing," *J. Acous. Soc. Amer.*, Vol. 27, (1955), p. 208, (Abstr.).

——, Harris, J. D. "Channel of Reception in Pitch Discrimination," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 124–131. (1845, 1955).

——. "The Diatonic Foundation of Hearing," *Acta Psychol.*, Vol. 11 (1955), pp. 432–445.

Pollack, I. "Sound Level Discrimination and Variation of Reference Testing Conditions," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 474–480. (E77504).

——. "Masking by a Periodically Interrupted Noise," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 353–355. (B30, 420, 1956).

Poulton, E. C. and Stevens, S. S. "On the Halving and Doubling of the Loudness of White Noise," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 329–331. (2512, 1955).

Quietsch, G. "Objektive und Subjektive Lautstärkemessungen," *Akust. Beih.*, Vol. 1 (1955), pp. 49–66.

Robinson, D. W. and Dadson, R. S. "A Re-determination of the Equal-Loudness Relations for Pure Tones," *Brit. J. Appl. Phys.*, Vol. 7 (1956), pp. 166–181. (A4280, 1956) (3271, 1956).

Rosenzweig, M. R. and Wyers, E. J. "Binaural Interaction at the Inferior Colliculi," *J. Comp. Physiol. Psychol.*, Vol. 48 (1955), pp. 426–431.

——, Rosenblith, W. A. "Responses to Successive Auditory Stimuli at the Cochlea and the Auditory Cortex," *Psychol. Monogr.*, Vol. 67, 1953.

Schäfer, O. and Schlitt, H. "Ein Beitrag zur Theorie des zweiohrigen Hörens," *Z. Angew. Phys.*, Vol. 7 (1955), pp. 165–168. (3137, 1955).

Schmidt, P. H. "Phantom Source Experiments in Auditory Localization," *Acta Physiol. Pharmacol. Neerl.*, Vol. 4 (1955), pp. 330–338.

——, Gemert, A. G. M., van, Vries, R. J., de, and Duyff, J. W. "Binaural Thresholds for Azimuth Difference," *Acta Physiol. Pharmacol. Neerl.*, Vol. 3 (1955), pp. 1–18.

Schubert, E. D. and Parker, C. D. "Addition to Cherry's Findings on Switching Speech between the Two Ears," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 792–794. (B29, 2220, 1955).

Shapley, J. L. "Reduction in the Loudness of a 250-cycle Tone in One Ear following the Introduction of Thermal Noise in the Opposite Ear," *Proc. Iowa. Acad. Sci.*, Vol. 61 (1954), pp. 417–422.

Snow, W. B. "Basic Principles of Stereophonic Sound," *IRE Trans.*, Vol. AU-3 (March-April, 1955), pp. 42–53. Abstr. Proc. IRE, Vol. 43 (April, 1955), p. 513.

Spieth, W. and Webster, S. C. "Listening to Differentially Filtered Competing Voice Messages," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 866–871. (E79807).

Steinbach, S. "Ein schnellarbeitender Suchtonverfahren für den Sprachfrequenzbereich," *Nachr. Tech.*, Vol. 6 (1956), pp. 396–400.

Stevens, S. S. "The Measurement of Loudness," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 815–829. (E79804).

——, Rogers, M. S., and Herrnstein, R. J. "The Apparent Reduction of Loudness: a Repeat Experiment," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 326–328. (B39, 423, 1956).

——, Loring, J. G. J., and Cohen, D. "Bibliography on Hearing," Cambridge: Harvard Univ. Press, 599 p. 1955. (B29, 5157, 1955).

——. "Calculation of the Loudness of Complex Noise," *J. Acous. Soc. Amer.*, Vol. 28 (1956), pp. 807–832.

Stott, A. and Axon P. E. "The Subjective Discrimination of Pitch and Amplitude Fluctuations in Recording Systems," *Proc. IEE*, B 102 (1955), pp. 643–656. (E79054) (3472, 1955).

Symmes, D., Chapman, C. F., and Halstead, W. C. "The Fusion of Intermittent White Noise," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 470–473. (E77508).

Thurlow, W. R. and Small, A. M. Jr. "Pitch Perception for Certain Periodic Auditory Stimuli," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 132–137. (1855, 1955).

Tweel, L. H., van der and Licklider, J. C. R. "Implication for Auditory Theory of Unitary Pitch Perception despite Diplacusis," *J. Acous. Soc. Amer.*, Vol. 28 (1956), p. 718.

Vermeulen, R. "Stereo Reverberation," *Philips Tech. Rev.*, Vol. 17 (1956), pp. 258–266. (A2762, 1956).

——. "A Comparison between Reproduced and 'Live' Music," *Philips Tech. Rev.*, Vol. 17 (1955), pp. 171–177.

Webster, J. C. and Sharpe, L. "Improvements in Message Reception resulting from 'Sequencing' Computing Messages," *J. Acous. Soc. Amer.*, Vol. 27 (1955), pp. 1194–1198. (A162, 1956).

Wertheimer, M. "The Variability of Auditory and Visual Absolute Thresholds in Time," *J. Gen. Psychol.*, Vol. 52 (1955), pp. 111–147. (B29, 2107, 1955).

Wever, E. G., Lawrence, M., and Bekesy, G., von. "A Note on Recent Developments in Auditory Theory," *Proc. Natl. Acad. Sci. Washington*, Vol. 40 (1954), pp. 508–512. (B29, 3515, 1955).

White, C. E. "How Do We Hear," *Audio*, Vol. 39, (1955), pp. 20–21, 57–58, (E79051).

Zwicker, E. "Der ungewöhnliche Amplitudengang der nichtlinearen Verzerrungen des Ohres," *Acustica, Akust. Beih.*, No. 1 (1955), pp. 67–74. (E78379).

——, Heinz, W. "Zur Häufigkeitsverteilung der menschlichen Hörschwelle," *Acustica, Akust. Beih.*, No. 1 (1955), pp. 75–80.

——. "Die Elementaren Grundlagen zur Bestimmung der Informations-kapazität des Gehörs," *Acustica*, Vol. 6 (1956), pp. 365–381.

c)

Armington, J. C. "Amplitude of Response and Relative Spectral Sensitivity of the Human Electroretinogram," *J. Opt. Soc. Amer.*, Vol. 45 (1955), pp. 1058–1064.

Attneave, F. "Some Informational Aspects of Visual Perception,"

Psychol. Rev., Vol. 61 (1954), pp. 183–193. (B29, 1960, 1955).

rlow, H. B. "Retinal Noise and Absolute Threshold," *J. Opt. Soc. Amer.*, Vol. 46 (1956), pp. 634–639. (A7910, 1956).

ck, L. H. "The Barnes-Czerny Apparatus for Demonstrating Quantal Fluctuations in Vision," *Amer. J. Psychol.*, Vol. 67 (1954), pp. 152–153. (B29, 50, 1955).

ersdorf, W. R. "Critical Duration in Visual Brightness Discrimination for Retinal Areas of Various Sizes," *J. Opt. Soc. Amer.*, Vol. 45 (1955), pp. 920–925, (B30, 5585, 1956).

llard, P. "Seuils Différentiels de Chromacité dans la Direction de la Primaire Bleue d'un système de synthèse Trichrome," *Rev. Opt.*, Vol. 34 (1955), pp. 371–405, (A850, 1956).

ttermann, M. E., Krauskopf, J. and Hockberg, J. E. "Threshold for Visual Form: A Diffusion Model," *Amer. J. Psychol.*, Vol. 67 (1954), pp. 205–219. (B29, 1962, 1955).

ounman, M. A. "Absolute Threshold Condition for Visual Perception," *J. Opt. Soc. Amer.*, Vol. 45 (1955), pp. 36–43.

ink, G., van den, and Bouman, M. A. "Variation of Integrative Actions in the Retinal System: an Adaptational Phenomenon," *J. Opt. Soc. Amer.*, Vol. 44 (1954), pp. 616–620. (B29, 3491, 1955).

own, R. H. *The Visual Discrimination of Velocity*, USNRL Rep. 1945, Nos. 4283, 4299, 4372. (B30, 5586, 5587, 5588, 1956).

ush, W. R. "Foveal Light Adaptation as Affected by the Spectral Composition of the Test and Adapting Stimuli," *J. Opt. Soc. Amer.*, Vol. 45 (1955), pp. 1047–1057.

opeland, A. H. "A Process Related to Certain Visual Phenomena," *Proc. Int. Congr. Math. (Amsterdam)* (1954), p. 284.

itchburn, R. W. and Fender, D. H. "The Stabilized Retinal Image," *Optica Acta*, Vol. 2 (1955), pp. 128–133. (A857, 1950).

——. "Eye movements in Relation to Retinal Action," *Optica Acta*, Vol. 1 (1954), pp. 171–177.

auberman, M. H. "Character Recognition for Business Machines," *Electronics*, Vol. 29 (1956), pp. 132–136.

ranit, R. *Receptors and Sensory Perception*, Oxford: Oxford Univ. Press, 369, 1955, Rev. *J. Opt. Soc. Amer.*, Vol. 45 (1955), p. 892.

——. "Sight and the Physiology of the Retina," *Acta Psychol. Fennica*, Vol. 1 (1951), pp. 21–29. (B29, 6667, 1955).

regory, R. L. "A Note on Summation Time of the Eye Indicated by Signal/Noise Discrimination," *Quart. J. Exp. Psychol.*, Vol. 7 (1955), pp. 147–148.

——. "An Experimental Treatment of Vision as an Information Source and Noisy Channel," *Proc. London Symp.*, (1955), pp. 287–302.

——, Cane, V. "A Statistical Information Theory of Visual Thresholds," *Nature*, Vol. 176 (1955), pp. 1272–1273. (E80210).

gelstam, E. and Johanson, L. "The Differential Sensitivity of the Eye to Colours in the Newtonian Scale Particularly for the Sensitive Colours," *Optica Acta*, Vol. 2 (1955), pp. 139–143. (In French) (A849, 1956).

ing-Ellison, P. and Jenkins, J. J. "The Durational Threshold of Visual Recognition as a Function of Word Frequency," *Amer. J. Psychol.*, Vol. 67 (1954), pp. 700–703. (B29, 2810, 1955).

ornfeld, M. "Resolving Power of the Eye," *Zh. Tekh. Fiz.*, Vol. 26 (1956), pp. 448–451. (In Russian) (A8538, 1956).

andis, C. "Determinants of the Critical Flicker-Fusion Threshold," *Physiol. Rev.*, Vol. 34 (1954), pp. 256–286.

ange, H. de, Dzn. "Experiments on Flicker and Some Calculations of an Electrical Analogue of the Foveal System," *Physica*, Vol. 18 (1952), pp. 935–950. (B29, 3448, 1955).

e Grand, Y. "Travaux Récents sur les Facteurs Physiques, Physiologiques et Psychologiques de la Vision des Détails," *J. Psychol. Norm. Path.* (1954), pp. 47–51, 69–78. (B29, 6689, 1955).

——. *Optique Physiologique III l'Espace Visuel*, 391 pp., 1956. (Ed. *Rev. d'Optique*).

nfoot, E. H. "Resolving Power and Information," *J. Opt. Soc. Amer.*, Vol. 46 (1956), p. 72.

ac Adam, D. L. "Rautian on Color Discrimination," *J. Opt. Soc. Amer.*, Vol. 45 (1955), pp. 1065–1071.

——. "Stereoscopic Perceptions of Size, Shape, Distance, and Direction," *J. Soc. Mot. Pict. Telev. Eng.*, Vol. 64 (1954), pp. 271–293.

ilne, L. J. and Milne, M. J. "Electrical Events in Vision," *Sci. Amer.*, Vol. 195 (1956), pp. 113–122.

orris, A., Katz, M. S. and Bowen, J. D. "Refinement of Checkerboard Targets for Measurement of Visual Acuity Limens," *J. Opt. Soc. Amer.*, Vol. 45 (1955), pp. 834–838.

owbray, G. H. and Gebhardt, J. W. "Differential Sensitivity of

the Eye to Intermittent White Light," *Science*, Vol. 121 (1955), pp. 173–175.

Mueller, C. G. "A Quantitative Theory of Visual Excitation for the Single Photo-Receptor," *Proc. Natl. Acad. Sci. Wash.*, Vol. 40 (1954), pp. 853–863. (B29, 5118, 1955).

——, and Berger, E. "Vision," *Ann. Rev. Psychol.*, Vol. 7 (1956), pp. 365–390. (B30, 5599, 1956).

Muller-Limmroth, H. W. "Zur Theorie des Farbensehens," *Die Naturwiss.*, Vol. 43 (1956), pp. 337–346, 364–370.

Piéron, H. "Le Problème du Rôle des Processus Récepteurs, Transmetteurs et Interpréteurs dans la Vision Chromatique," *Acta Psychol.* (Abstr.) Vol. 11 (1955), pp. 220–221.

Pirenne, M. H. and Marriott, F. H. "Absolute Threshold and Frequency of Seeing Curves," *J. Opt. Soc. Amer.*, Vol. 45 (1955), pp. 909–912. (A10141, 1955).

——. "Quantum Theories of the Absolute Threshold, The Influence of Retinal Position and Area," *Optica Acta*, Vol. 1 (1954), pp. 151–155.

Pollack, Ph. "A Note on Monocular Depth Perception," *Amer. J. Psych.*, Vol. 68 (1955), pp. 315–318.

Polyak, S. *The Vertebrate Visual System: Its Origin, Structure and Function and its Manifestations in Disease with an Analysis of its Rôle in the Life of Animals and in the Origin of Man*, Chicago: Univ. of Chicago Press, 1600 pp.

Popov, C. "Sur l'accroissement de la Sensation Lumineuse par la Stimulation Sonore Etudiée par la Méthode Electroencéphalographique et par la Méthode des Images Consécutives," *C. R. Acad. Sci., Paris*, Vol. 240 (1955), pp. 1268–1271.

Rady, A. A. and Ishak, I. G. H. "Relative Contribution of Disparity and Convergence to Stereoscopic Acuity," *J. Opt. Soc. Amer.*, Vol. 45 (1955), pp. 530–534. (E77453).

Rapoport, A. "Application of Information Networks to a Theory of Vision," *Bull. Match. Biophysics*, Vol. 19 (1955), pp. 15–33.

Rautian, G. N. "Thresholds of Colour Discrimination," *C. R. Acad. Sci. U.S.S.R.*, Vol. 92 (1952), pp. 943–946. (In Russian).

Schade, O. H. Sr. "Optical and Photoelectric Analog of the Eye," *J. Opt. Soc. Amer.*, Vol. 46 (1956), pp. 721–739. (E88430).

Schelling, H. von. "Indication for the Existence of at least Four Base Excitations in Color Vision," *J. Opt. Soc. Amer.*, Vol. 45 (1955), pp. 209–215. (B29, 6713, 1955).

Sperry, R. W. and Schneider, H. "The Eye and the Brain," *Sci. Amer.*, Vol. 194 (May, 1956), pp. 48–68.

Stevens, S. S. "Decibels of Light and Sound," *Phys. Today*, Vol. 8 (1955), pp. 12–17. (E79199).

Sziklay, G. C. "Some Studies in the Speed of Visual Perception," IRE TRANS., Vol. IT-2 (September, 1956), pp. 125–128. Abstr. PROC. IRE, Vol. 44 (November, 1956), p. 1643.

Tanner, W. P. and Swets, J. A. "A Decision-Making Theory of Visual Detection," *Psych. Rev.*, Vol. 61 (1954), pp. 401–409. (B29, 5133, 1955).

Thomas, G. J. "Vision," *Ann. Rev. Psychol.*, Vol. 6 (1955), pp. 63–94. (B29, 5135, 1955).

Teichner, W. H., Kobrick, J. L., and Dusek, E. R. "Commonplace Viewing and Depth Discrimination," *J. Opt. Soc. Amer.*, Vol. 45 (1955), pp. 913–920. (B30, 5606, 1956).

Weale, R. A. "The Absolute Threshold of Vision," *Physiol. Rev.*, Vol. 35 (1955), pp. 233–246.

Weymouth, F. W. "Visual Acuity—An Analysis of the Stimulus Situations," *Amer. J. Optom.*, Vol. 32 (1955), pp. 507–519.

Willmer, E. N. "The Physiology of Vision," *Ann. Rev. Physiol.*, Vol. 17 (1955), pp. 339–366.

d)

Bar-Hillel, Y. "Can Translation be Mechanized," *Methodos*, Vol. 7 (1955), pp. 45–62.

Barnard, G. A. "Statistical Calculation of Word Entropies for Four Western Languages," IRE TRANS., Vol. IT-1 (March, 1955), pp. 49–53. Abstr. PROC. IRE, Vol. 43 (July, 1955), p. 893.

Bell, D. A. and Ross, A. S. C. "Negative Entropy of Welsh Words," *Proc. London Symp.* (1955), pp. 149–153.

Booth, A. D. "Use of a Computing Machine as a Mechanical Dictionary," *Nature*, Vol. 176 (1955), p. 565. (E78546).

Booth, A. D. "Calculating Machines and Mechanical Translation," *Discovery*, Vol. 15 (1954), pp. 280–285. (B29, 940, 1955).

——. "The Influence of Context on Translation," *Inform. Theory, London Symp.* (1955), pp. 181–183.

Burton, N. G. and Licklider, J. C. R. "Long-Range Constraints in the Statistical Structure of Printed English."

Ceccato, S. and Maretti, E. "Suggestions for Mechanical Translation," *Inform. Theory, London Symp.* (1955), pp. 171–180.

Chaitanya, D. "An Examination of the Nature of Structure," *Nachr. Tech. Fachber.*, Vol. III (1956), pp. 63–72.

Chomsky, A. N. "Three Models for the Description of Language," IRE TRANS., Vol. IT-2 (September, 1956), pp. 113–124.

Cleave, J. P. and Zacharov, B. "Language Translation by Electronics," *Wireless World*, Vol. 61 (1955), pp. 433–435. (E79188).

——. "A Program for Braille Translation," *Inform. Theory, London Symp.* (1955), pp. 184–194.

Fairthorne, R. A. "Some Clerical Operations and Languages," *Inform. Theory, London Symp.* (1955), pp. 111–120.

Fant, G. *Relativa Förekomsten av Ord och Talljud i Svenska Språket*, R.V.K., 54, 8 pp.

Fano, R. M. *Entropy of Printed English*. Quart. Prog. Rep. M.I.T. Res. Lab. Electronics (April, 1955), pp. 47–48.

Fucks, W. "Mathematical Theory of Word Formation," *Inform. Theory, London Symp.* (1955), pp. 154–170.

——. *Mathematische Analyse von Sprachelementen, Sprachstil und Sprachen. Arbeitsgem. Forschung Londes Nordrhein-Westfalen Heft 34a*, 110 pp.

——. "Die Mathematische Gesetze der Bildung von Sprachelementen aus ihren Bestandteilen. Theorie der Wordbildung aus Silben und der Silbenbildung aus Lauten," *Nachr. Tech. Fachber*, Vol. III (1956), pp. 7–62.

Gemelli, A., Sacerdote, G. G., and Bellusi, G. "Nuovi Contributi Electroacustica alla studio del canto," *Boll. Soc. Ital. Fonet. Sperim*, Vol. VI (1956), 48 pp.

Guiraud, P. "Le Substrat Informationnel de la Sémantisation," *Bull Soc. Linguistique*, Vol. 50 (1954), pp. 119–133.

Halle, M. "The Strategy of Phonemics," *Word*, Vol. 10 (1954), pp. 197–209.

Husson, R. "Structure Aucostique et Génèse Physiologique des Sons du Langage," *Rev. Gén. Sci. Pures Appl.*, Vol. 63 (1956), pp. 159–178. (E85751).

Josselson, H. H. *The Russian Word Count and Frequency Analysis of Grammatical Categories of Standard Literary Russian*, Wayne Univ. Press, 274 pp. 1953. (B29, 3966, 1955).

Kaizer, L. "A Statistical Research of Spoken Language, (Dutch)," *Orbis*, Vol. 4 (1955), pp. 237–248.

Kaplan, A. "An Experimental Study of Ambiguity and Context," *Mech. Transl.*, Vol. 2 (1955), pp. 39–46.

Lisman, J. H. C. "Statistische analyse van muziek," *Statistica*, Vol. 8 (1955), pp. 83–91.

——. "Enkele Frequentieverdelingen in de Muziek," *Statistica*, Vol. 2 (1948), pp. 120–123.

Locke, W. N. "Translation by Machine," *Sci. Amer.*, Vol. 194 (1956), pp. 29–33.

—— and Booth, A. D. *Machine Translation of Languages*, Wiley Tech. Press, 243 pp. 1955. (E-L3734).

Mandelbrot, B. "A Law of Berry and the Definition of Stress," (distributed at London Symp. 1955).

——. "Théories du comportement. Une définition de la Cybernétique; applications linguistique," *Rev. Gén. Sci.*, Vol. 52 (1955), pp. 278–294.

Meredith, G. P. "Language, Meaning and Mind," *Nature*, Vol. 176 (1955), pp. 673–674. (E79052).

Meyer-Eppler, W. "Physikalische Analogien, linguistische Strukturen," *Phys. Blätter*, Vol. 11 (1955), pp. 445 449. (E79809).

——. "Elektronische Musik (Ihre stofflichen undinformationstheoretischen Grundlagen)," aus: Klangstrucktur der Musik, pp. 135–138. Verlag Radio-Foto-Kinotechnik, Berlin, Borsigwalde, 1955.

Michel, G. A. "Ueber ein Masz für die Buchstabenüberladung der Natürlichen Sprachen," *Frequenz*, Vol. 10 (1956), pp. 383–384.

Miller, G. A. *Psycholinguistics*. (In M. Lindsey: *Handbook Social Psychol.*) Vol. II, pp. 693–708.

——. "Information Theory and the Study of Speech," *Current Trends*, pp. 119–139. (B29, 835, 1955).

Pinkerton, R. C. "Information Theory and Melody," *Sci. Amer.*, Vol. 194, No. 2 (1956) pp. 77–87.

Pol, Balth. van der. "An Iterative Translation Test. Information Theory," *Proc. London Symp.*, Abstr. (1955), pp. 397–398.

Reifler, E. "Mechanical Demonstration of the Constituents of German Substantive Compounds," *Mech. Transl.*, Vol. 2 (1955), pp. 3–14.

Sankaran, G. "Structure in Speech; The Physical Reality of th Phoneme," *Nachr. Tech. Fachber*, Vol. III (1956), pp. 56–63.

Varsharskiĭ, L. A. and Litvak, I. M. "Investigation of Forman Composition and Several Other Physical Characteristics of th Sound of Russian Speech, *Prob. Physiol. Acous. U.S.S.R.*, Vol (1955).

Williams, C. B. "Studies in the History of Probability and Statistic IV, A Note on an Early Statistical Study of Literary Style, *Biometrika*, Vol. 43 (1956), pp. 248–256.

Wilson, K. V. N. *Information Transmission and Optical Codin in Natural Language Messages*. Ph.D. Thesis, Univ. of Illinoi 1954. *Diss. Abstr.*, Vol. 14 (1954), pp. 1802–1803. (B29, 860 1955).

Yngve, V. H. "The Translation of Languages by Machine," *Inform Theory, Proc. London Symp.* (1955), pp. 195–205.

——. "Sentence for Sentence Translation," *Mech. Transl.*, Vo 2 (1955), pp. 29–37.

——. "Gap Analysis and Syntax," IRE TRANS., Vol. IT-2 (Septem ber, 1956), pp. 106–112. Abstr. PROC. IRE, Vol. 44 (November 1956), p. 1643.

VII. a) OTHER BIOLOGICAL APPLICATIONS (CYBERNETICS AND THE NERVOUS SYSTEM); b) HUMAN ENGINEERING c) GROUP COMMUNICATION AND LEARNING.

a)

Allanson, J. T. "Some Properties of a Randomly Connected Neura Network," *Inform. Theory, Proc. London Symp.* (1955), pp 303–313.

Anon. "The Artificial Heart and Lung Machine," *Electronic Eng* Vol. 28 (1956), p. 550.

Ashby, W. R. *An Introduction to Cybernetics*, Chapman Hall 297 pp., 1956. Rev. *Electronic and Radio Eng.*, Vol. 34 (1957 p. 34.

Barlow, J. S. and Brown, R. M. *An Analog Correlator System fo Brain Potentials*, M.I.T. Res. Lab. Electronics Tech. Rep. 300 1955.

—— and Brazier, M. A. B. *Correlation Studies of Electroencephalo grams*, Quart. Prog. Rep. M.I.T. Res. Lab. Electronics (January 1956), pp. 141–145.

Blackwell, H. R. *Psychophysical Thresholds: Experimental Studie of Methods of Measurement*, University of Michigan Eng. Res Inst. Bull. No. 36, 227 pp., 1953. (B29, 1746, 1955).

Bouthillon, L. "Philosophie des Calculateurs Automatiques," *Onde Elect.*, Vol. 36 (1956), pp. 693–708.

Butler, J. A. V. "Mind and Matter, a Monist View," *Pengui Sci. News*, Vol. 34 (1954), pp. 95–102.

Castillo, J. del and Katz, B. "Biophysical Aspects of Neuro-Muscula Transmission," *Prog. in Biophys.*, Vol. 6 (1956), pp. 122–170.

Cohn, D. L. "Optimal Systems. I, II The Vascular System, II Muscle," *Bull. Math. Biophys.*, Vol. 16 (1954), pp. 59–74, Vol. 1 (1955), pp. 219–227, 309–329.

Cole, K. S. "Electro-Ionics of Nerve Action," IRE TRANS., Vol PGME-6, (January, 1956), pp. 28–48.

Cossa, P. "La Cybernétique 'Du Cerveau Humain aux Cervau artificiels,'" *Coll. "Evolution des sciences*,"* Vol. 4, Masson, 1955.

Couffignal, L. "Machines á calculer électroniques et cybernétique,' *Conv. Elettronica Telev., Milano*, Vol. 2 (1954), pp. 1085–1095.

Cuzzer, A. "La Cibernetica come Teoria dell 'Informazione, *Conv. Elettronica Telev., Milano*, Vol. 2 (1954), pp. 1164–1169.

Delpech, L. "Psychanalyse et cybernétique," *Proc. XIth In Congr. Phil.*, Vol. 7 (1953), pp. 155–161. (B29, 1722, 1955).

Deutsch, J. A. "A Machine with Insight," *Quart. J. Exp. Psychol* Vol. 6 (1954), pp. 6–11. (B29, 1723, 1955).

——. "A Theory of Shape Recognition," *Brit. J. Psych.*, Vol. 4 (1955), pp. 30–37. (B29, 6661, 1955).

Doyon, A. and Liaigre, L. "La Vocation de Vaucanson ou d Totémisme à la cybernétique," *Ann. Fr. Chronom.*, Vol. 3 (1956 pp. 1315–1326. (E87657).

Fack, H. "Die Impulsübertragung im Nervensystem," *Imp. t.*, pp 259–288. See Winckel XI, p. 46.

Fitz Hugh, R. "Mathematical Models of Threshold Phenomena i the Nerve Membrane," *Bull. Math. Biophys.*, Vol. 17 (1955), pp 257–278.

Franck, U. F. "Models for Biological Excitation Processes," *Proc Biophys.*, Vol. 6 (1956), pp. 171–206.

Frankel, S. "On the Design of Automata and the Interpretation c Cerebral Behavior," *Psychometrika*, Vol. 20 (1955), pp. 149–162.

udenthal, H. "Machines Pensantes," *Conférence fait au Palais
e la Découverte*, Vol. 9 (May, 1953), Serie D, Univ. de Paris, 16 pp.

nov, G. "Information Transfer in the Living Cell," *Sci. Amer.*,
ol. 193 (1955), pp. 70–79.

dberg, S. "Probability Models in Biology and Engineering,"
. *Soc. Ind. Appl. Math.*, Vol. 2 (1954), pp. 10–19.

nenberg, E. "Thinking Machines and Human Personality,"
Comp. Aut., Vol. 4 (1955), pp. 6–9.

illaume, P. "Cybernétique et Psychologie. II Machines et
Systèmes Physiques," *J. Psychol. Norm. Path.*, (1954), pp. 47–51,
360–378, 486–499. (B29, 8167, 1955).

lman, H. E. "Electrocardiography as a Problem for Modern
Engineering," IRE TRANS., Vol. PGME-1 (November, 1953),
pp. 1–26.

xley, H. E. "Muscular Contraction," *Endeavour*, Vol. 15 (1956),
pp. 177–186.

rreman, G. "Topological Information Content and Chemical
Reactions," *Bull. Math. Biophys.*, Vol. 17 (1956), pp. 279–286.

tsoff, L. O. "Brains, Thinking and Machines," *Methodos*, Vol. 6
(1955), pp. 279–286.

eene, S. C. *Representation of Events in Nerve Nets and Finite
Automata, Automata Studies*, 3–41, Ann. Math. Studies No. 34,
Princeton Univ. Press, 1956.

etsky, E. J. "Ultrastability and the Delft Homeostat," *Tijdschr.
Ned. Radiogenootschap*, Vol. 21 (1956), pp. 265–280.

nakau, C. E. T. "Frequency Analysis of Neuronal Time Series,"
Trans. Roy. Physiograph. Soc. Lond. Bd. 26, (1956), 20 pp.

ndsley, D. B. "Higher Functions of the Central Nervous
System," *Ann. Rev. Physiol.*, Vol. 17 (1955), pp. 311–338.

ac Kay, D. M. "On Comparing the Brain with Machines," *Amer.
Scientist*, Vol. 42 (1954), pp. 261–268. *Scient. J. Royal Coll. Sci.*,
Vol. 24 (1954), pp. 97–105.

alrieu, Ph. *Les Origines de la Conscience du Temps*. Paris Presses
Univ. de France, 1953, 158 pp., (B29, 1947, 1955).

acy, J. *A Probability Model for Cortical Response to Successive
Auditory Clicks*. Quart. Prog. Rep. M.I.T. Res. Lab. Electronics
(July, 1954), pp. 75–80.

cCulloch, W. S. "Information in the Head," *Current Trends*,
pp. 92–118. (B29, 200, 1955).

eerloo, J. A. M. van. *The Two Faces of Man: Two Studies on the
Sense of Time and on Ambivalence*. New York: Intern. Univ.
Press, 237 pp., 1954. (B29, 3235, 1955).

ewell, A. and Simon, H. A. "The Logic Theory Machine," IRE
TRANS., Vol. IT-2 (September, 1956), pp. 61–79. Abstr. PROC.
IRE, Vol. 44 (November, 1956), p. 1643.

dfield, R. C. "Apparent Fluctuations of a Sensory Threshold,"
Quart. J. Exp. Psychol., Vol. 7 (1955), pp. 101–115. (B30, 4002,
1956).

eiffer, J. *The Human Brain*. New York: Harper, 273 pp., 1955.

ecker, J. A. "Toward a Theoretical Brain-Model," *J. Pers.*, Vol.
22 (1954), pp. 310–325. (B29, 41, 1955).

osser, C. L. "Comparative Physiology of Nervous Systems and
Sense," *Organs. Ann. Rev. Physiol.*, Vol. 16 (1954), pp. 103–124.
(B29, 206, 1955).

lter, K. *A Time-Gated Amplitude Quantizer for Neural Signals.*
M.I.T. Res. Lab. Electronics Tech. Rep. 275, 1954.

poport, A. "Technological Models of the Nervous System,"
Methodos, Vol. 7 (1955), pp. 131–146.

shevsky, N. "Life, Information Theory and Topology," *Bull.
Math. Biophys.*, Vol. 17 (1955), pp. 229–235.

chester, N., Holland, J. H., Haibt, L. H., and Duda, W. L.
"Tests of a Cell Assembly Theory of the Action of the Brain,
Using a Large Digital Computer," IRE TRANS., Vol. IT-2
(September, 1956), pp. 80–93.

rton, G. "The Astral Religion of Antiquity and the 'Thinking
Machine' of To-day," *Vistas in Astronomy*, Vol. 1, pp. 51–60,
Pergamon Press, 1955.

hottlaender, S. "Ueber eine analytische Methode zur Unter-
suchung automatisch gesteuerten Bewegungen," *Proc. Congr.
Math. Amsterdam*, (1954), pp. 170–171.

menzi, V. "Can Induction be Mechanized," *Methodos*, Vol. 7
(1955), pp. 147–151. *Inform. Theory, Proc. London Symp.* (1955),
pp. 226–230.

fanelli, A. "New Aspects and Considerations on Nerve Cell
Cytomorphose," *La Ricercha Scientifica*, Vol. 25 (1955), pp.
2778–2795.

nner, W. P. "Theory of Recognition," *J. Acous. Soc. Amer.*,
Vol. 28 (1956), pp. 882–889.

Taylor, W. K. "The Electrical Simulation of Some Nervous Systems
Functional Activities," *Inform. Theory, Proc. London Symp.*
(1955), pp. 314–328.

Tournay, A. "Recherches nouvelles sur les Régulations et les
Ajustements par le Système Nerveux," *J. Psychol. Norm. Path.*
(1954), pp. 20–42, 47–51. (B29, 6613, 1955).

Tustin, A. "Cybernetics and All That," *J. IEE*, Vol. 1 (1955),
pp. 634–635. (E80024).

Vries, Hl. de. "Physical Aspects of Sense Organs," *Prog. in Bio-
phys*, Vol. 6 (1956), pp. 207–264.

Wagner, R. *Probleme und Beispiele biologischer Regelung*. Stuttgart:
Thieme, 219 pp., 1954. *Rev. Sci.*, Vol. 122 (1955), p. 38.

Wall, P. D., Lettvin, J. Y., McCulloch, W. S., and Pitts, W. H.
"Factors Determining the Channel Capacities and Design
Constraints in Groups of Nerve Fibers," 1956 IRE CONVENTION
RECORD, Part 4, pp. 128–131.

——. "Factors Limiting the Maximum Impulse Transmitting
Ability of an Afferent System of Nerve Fibers," *Inform. Theory,
Proc. London Symp.* (1955), pp. 329–344.

Wyss, O. A. M. "Respiratory Centre and Reflex Control of Breath-
ing," *Helv. Physiol. Pharmacol. Acta* Suppl. X, (1954), 33 pp.

b)

Augenstine, L., Blank, A. A., Quastler, H., and Wayner, M. *Human
Performance in Information Transmission*, Tech. Rep. Control
Syst. Lab., Univ. of Illinois, 1955.

Bricker, P. D. *Information Measurement and Reaction Time*. A
review (in "Information Theory in Psychology, Problems and
Methods") see Quastler.

Cherry, E. C. *On the Validity of Application of Communication
Theory to Human Operator Problems*, Directorate of Weapons
Research (Defense) WR (D) Rep., pp. 40–53, February, 1956.

Christie, L. S. and Luce, R. D. "Decision Structure and Time
Relations in Simple Choice Behavior," *Bull. Math. Biophys.*,
Vol. 18 (1956), pp. 89–112. (C1221, 1956).

Crossman, E. R. F. W. *The Information-Capacity of the Human
Operator in Symbolic and Non-Symbolic Control Processes*. WR(D)
Rep., pp. 4–18.

——. "Entropy and Choice-Time; the Effect of Frequency Un-
balance on Choice-Response," *Quart. J. Exp. Psych.*, Vol. 5
(1953), pp. 41–51.

——. "Perceptual Activity in Manual Work," *Research*, Vol. 9
(1955), pp. 42–49.

Deininger, R. L. and Fitts, P. M. *Stimulus-Response Compatibility
Information Theory and Perceptual Motor Performance.* (in:
Information Theory in Psychology) (See Quastler).

Draper, J., (ed.). *The Application of Information Theory to Human
Operator Problems*, WR(D) Rep. Min. of Supply, 92 pp., Feb-
ruary, 1956.

Fitts, P. M. "The Information Capacity of the Human Motor
System in Controlling the Amplitude of Movement," *J. Exp.
Psychol.*, Vol. 47 (1954), pp. 381–391.

Fogel, L. J. "An Analysis for Human Flight Control," PROC. IRE.
Abstr., Vol. 44 (March, 1956), p. 390.

Freeman, J. T. and Engler, J. "Perceptual Recognition Thresholds
as a Function of Multiple and Single Set and Frequency of
Usage of the Stimulus Material," *Percept. Mot. Skills*, Vol. 5
(1955), pp. 149–154. (B30, 5593, 1956).

Hick, W. E. *Some Miscellanea on Information Theory and the Human
Operator*. WR(D) Rep. pp. 75–87 (1956), (See Draper).

——. "On the Rate of Gain of Information," *Quart. J. Exp. Psychol.*
Vol. 4 (1952), pp. 11–26.

Hirsch, J. "Communication by Vibrating Tactile Stimuli," IRE
TRANS., Vol. PGME-7 (1956), pp. 29–37.

Hoberman, M. "A Design-Philosophy for Man-Machine Control
Systems," PROC, IRE, Vol. 43 (May, 1955), p. 623.

Holding, D. H. *Information Theory in Psychology*. WR(D) Rep.,
pp. 54–74 (1956) (See Draper).

Hyman, R. "Stimulus Information as a Determinant of Reaction
Time," *J. Exp. Psychol.*, Vol. 45 (1953), pp. 188–196.

Kemeny, J. G. "Man Viewed as a Machine," *Sci. Amer.*, Vol. 192
(1955), pp. 58–67.

Knowles, W. B., Garvey, W. D. and Newlin, E. P. "The Effect of
Speed and Load on Display Control Relationships," *J. Exp.
Psychol.*, Vol. 46 (1953), pp. 65–75.

Krendel, E. S. "Frequency Analysis of Human Responses," *J.
Amer. Stat. Assoc.* Abstr., Vol. 51 (1956), p. 514.

Mackworth, N. H. "Vigilance," *Nature*, Vol. 178 (1956), pp. 1375–1377.

Miller, G. A. "Human Memory and the Storage of Information," IRE TRANS., Vol. IT-2 (September, 1956), pp. 129–137. Abstr. PROC. IRE, Vol. 44 (November, 1956), p. 1644.

North, J. D. "Application of Communication Theory to the Human Operator," WR(D) Rep., pp. 19–39, February, 1956 (See Draper).

——. "Application of Communication Theory to the Human Operator," *Inform. Theory, Proc. London Symp.*, (1955), pp. 372–396.

Pierce, J. R. and Karlin, J. E. "Maximum Information Rate through a Human Channel in Reading," Abstr. *Science*, Vol. 122 (1955), p. 879.

Quastler, H. "Studies of Human Channel Capacity," *Inform. Theory, Proc. London Symp.* (1955), pp. 361–371.

——, (ed.). *Information Theory in Psychology*. Glencoe, Ill.: Free Press, 1955.

Rapoport, A. "On the Application of the Information Concept to Learning Theory," *Bull. Math. Biophys.*, Vol. 18 (1956), pp. 317–323.

Rogers, C. R. and Skinner, B. F. "Some Issues Concerning the Control of Human Behavior," *Science*, Vol. 124 (1956), pp. 1057–1066.

Strand, J. M. "A Simulation System for the Study of Human Dynamics," *Proc. Natl. Simul. Conf.*, Dallas, 1956.

Zaremba, S. K. "Remarks on Feedback and Open-Chain Following," *Appl. Inf. Theory*. WR(D) Rep. pp. 33–36, February, 1956 (See Draper).

c)

Arrow, K. J. "Social Choice and Individual Values," *Cowles Comm. Mon.*, No. 12, 99 pp. 1951. New York: Wiley.

Bales, R. F. "How People Interact in Conferences," *Sci. Amer.*, Vol. 192 (1955), pp. 31–35.

Block, J. and Bennett, L. "The Assessment of Communication: Perception and Transmission as a Function of the Social Situation," *Hum. Rel.*, Vol. 8 (1955), pp. 317–325. (B30, 5927, 1956).

Broadbent, D. A. "The Concept of Capacity and the Theory of Behaviour," *Inform. Theory, Proc. London Symp.* (1955), pp. 354–360.

Bush, R. R. and Mosteller, F. *Stochastic Models for Learning*, New York: Wiley, 365 pp., 1955. *Rev. Sci.*, Vol. 122 (1955), p. 1023. (C1136, 1955).

Cherry, E. C. "On the Mathematics of Social Communications," *Int. Soc. Sci. Bull.*, Vol. VI No. 4, pp. 609–621. Unesco, Paris.

Christie, L. S. "Organization and Information Handling in Task Groups," *J. Oper. Res. Soc. Amer.*, Vol. 2 (1954), pp. 188–196. (B29, 3959, 1955).

Frisch, K. von. *The Dancing Bees*. New York: Harcourt-Brace, 183 pp., 1955. *Rev. Sci.*, Vol. 122 (1955), p. 379.

—— and Lindauer, M. "Ueber die Fluggeschwindigkeit der Bienen und über ihre Richtungsweisung bei Seitenwind," *Naturwiss.*, Vol. 13 (1955), pp. 377–385.

Harlow, H. F. "The Brain and Learned Behaviour," *Comp. Aut.*, Vol. 4 (1955), pp. 6–14.

Landau, H. G. "The Distribution of Completion Times for Random Communication in a Task-Oriented Group," *Bull. Math. Biophys.*, Vol. 16 (1954), pp. 187–202.

Lazarsfeld, P. (ed.). *Mathematical Thinking in the Social Sciences*. Glencoe, Ill.: Free Press, *Rev. Bull. Math. Biophys.*, Vol. 17 (1955), pp. 237–241.

Licklider, J. C. R. (ed.). *Problems in Human Communication and Control*. Cambridge, Mass., 1955, 203 pp.

Pollack, I. "Assimilation of Sequentially Encoded Information," *Amer. J. Psych.*, Vol. 66 (1953), p. 421.

Selfridge, O. G. "Pattern Recognition and Learning," *Inform. Theory, Proc. London Symp.* (1955), pp. 345–353.

Shimbel, A. "Structure in Communication Nets," *Proc. Symp. Inform. Networks*, Polytech. Inst. Brooklyn (1954), pp. 199–203.

VIII. TELEVISION

Balakrishnan, A. A. "Multipath Distortion of TV Signals and the Design of a Corrective Filter," 1956 IRE CONVENTION RECORD, Part 4, pp. 167–168.

Baldwin, M. W. and Nielsen, G. "The Subjective Sharpness of Simulated Color Television Pictures," *J. Opt. Soc. Amer.*, Vol.

46 (1956), pp. 681–685. (E88095). IRE TRANS., Vol. BTS-4 (March, 1956), pp. 17–21. (E86292).

Baugh, C. W. Jr. and Sweeney, H. E. "Transient Response v Chrominance Bandwidth in Simultaneous Color Televisio Receivers," 1956 IRE CONVENTION RECORD, Part 3, pp. 77–83

Baumgardt, E. "Vision et télévision. quelques problèmes de physio logic sensorielle," *Cah. Et. Radio. Télév.* No. 1 (1954), pp. 80–98 (E77254).

Borenemann, I. "Wahl von Farbbild-Luminophoren und auftretend Farbinformations-Verluste," *Nachr. Tech.*, Vol. 6 (1956), pp 534–537.

Boutry, G. A. "L'Oeil Humain devant le Spectacle de Télévision, *Optica Acta*, Vol. 2 (1955), pp. 81–100.

——, Billard, P., and Le Blan, L. "Le Comportement de l'Oeil e la Transmission Rationelle d'Images Simultanées. III. *Ond Elect.*, Vol. 35 (1955), pp. 5–21. (2118, 1955).

Bryan, J. S., Clapp, R. G., Creamer, E. M., Moulton, S. W. an Partin, M. E. "A New Color Television Display—The Appl System," PROC. IRE. Abstr., Vol. 44 (March, 1956), p. 406.

Capon, J. *Bounds to the Entropy of Television Signals* M. Sc. Thesi Dept. Elect. Eng. M.I.T., 1955.

——. *Bounds to the Entropy of Television Signals*. Quart. Prog Rep. M.I.T. Res. Lab. Electronics (April, 1955), pp. 49–51.

Clapp, R. G., Creamer, E. M., Moulton, S. W., Partin, M. E. an Bryan, J. S. "A New Beam-Indexing Color Television Displa System," PROC. IRE, Vol. 44 (1956), pp. 1108–1114. 1956 IRI CONVENTION RECORD, Part 3, pp. 94–100.

Deutsch, S. "The Possibilities of Reduced Television Bandwidth, Ire TRANS., Vol. BTR-2, (October, 1956) pp. 69–81.

Fortuin, G. J. "Visual Acuity in Connection with Television, *Philips Tech. Rev.*, Vol. 16 (1954), pp. 172–178. (D2979, 1955).

Funk, S. "Gradationsfragen im Fernsehprozess," *Nachr. Tech* Vol. 6 (1956), pp. 114–120.

Gibson, W. G. and Schroeder, A. C. "Color Television Luminanc Detail Rendition," PROC. IRE, Vol. 43 (August, 1955), p 918–923.

Gouriet, G. G. "Bandwidth Reduction in Television," *E. B. U Bull. 7* No. 35 (1956), pp. 15–26. *J. Telev. Soc.*, Vol. 8 (1956 pp. 103–108.

Gurfinkel, B. B. "Television Systems with Statistical Correlation, *Radiotehnika and Electronics*, (Moscow), Vol. 1 (1956), pp. 478 496. (In Russian.)

Haantjes, J. and Teer, K. "Compatible Colour-Television," *Wireles Eng.*, Vol. 33 (1956), pp. 3–9, 39–46.

Kovasznay, L. S. G. and Joseph, H. M. "Image Processing, PROC. IRE, Vol. 43 (June, 1955), pp. 560–570. (E76948, 2455 1955).

Kretzmer, E. R. "Reduced Alphabet Representation of Televisio Signals," 1956 IRE CONVENTION RECORD, Part 4, pp. 140–147

——. "Versuche zur günstiger Codierung von Bild Information, *Nachr. Tech. Fachber.*, Vol. 6 (1956), pp. 19–25.

Lagowitz, G. "Ueber Entropieverluste in Fernsehempfägern, *Nachr. Tech.*, Vol. 5 (1956), pp. 396–399. (E79350).

Mac Adam, D. L. "Perception of Colors in Projected and Televise Pictures," *Soc. Mot. Pict. Telev. Eng.*, Vol. 65 (1956), pp. 455–469

Maurice, R. D. A., Gilbert, M., Newell, G. F., and Spencer, J. C "The Visibility of Noise in Television," *BBC Eng. Div. Mo* No. 3 (1955), p. 5.

Murakami, T. and Corrington, M. S. "Application of the Fourie Integral in the Analysis of Color Television Systems," IR. TRANS., Vol. CT-2 (December, 1955), pp. 250–255.

Neidhart, P. "Der Informationstheoretische Wirkungsgrad (Wiede gabefaktor) einer Fernsehkette," *Nachr. Tech. Fachber.*, Vol. (1956), pp. 56–63.

——. "Korrelation, Redundanz, und Rauschen im Fernsehsignal, *Nachr. Tech.*, Vol. 5 (1955), pp. 341–346. (E77873).

——. "Informationsinhalt und Frequenzbandbreite des Chrominan Signalsim Farbfernsehverfahren mit einem Zwischenträger, *Nachr. Tech.*, Vol. 6 (1956), pp. 529–533.

Poincelot, P. "Les Régimes Transitoires en Télévision," *Ond Elect.*, Vol. 35 (1955), pp. 764–767. (E77806).

Richman, D. "Directions of Improvement in N.T.S.C. Colc Television Systems," PROC. IRE, Vol. 44 (September, 1956 pp. 1125–1139.

Roschke, E. M., Druz, W. S., Eilers, C., and Pulles, J. "An Integrate System of Coded Picture Transmission," 1956 IRE CONVENTIO RECORD, Part 7, pp. 173–179.

lling, H. von. "Advanced Color Geometry," *J. Opt. Soc. Amer.*, ol. 45 (1955), pp. 1072–1079.

ber, H. "Informationstheorie in Optik und Fernsehen," *Optik*, ol. 13 (1956), pp. 350–364.

enfelder, H. "Zur These von Mertz und Gray," *Frequenz*, Vol. 10 956), pp. 142–147. (E85518).

reiber, W. F. "The Measurement of Third Order Probability istributions of Television Signals," *IRE Trans.*, Vol. IT-2 eptember, 1956), pp. 94–105. Abstr. Proc. IRE, Vol. 44 November, 1956), p. 1643.

roeter, F. "Möglichkeiten der Verminderung des Frequenzband- ufwandes beim Teleran System," *Elektron. Rdschau*, Vol. 9 955), p. 272. (E78178).

-. "Réflexions sur l'evolution future de la Télévision," *La icerca Scientifica*, Vol. 7 (1954), pp. 1613–1639.

er, A. J. "Bandwidth Reduction in Television Relaying," roc. IRE Austr., Vol. 16 (1955), pp. 218–225, 261–276. (E78060, 78797).

b, T. "Matrixing and Encoding Color for Telecasting," *Elec- onics* (June 29, 1956), 6A, R36–R41.

ile, R. "Zur Definition des durch die statistischen Schwankungen estimmten Störabstandes in Fernsehen," *Arch. elektr. Uebertr.*, ol. 10 (1956), pp. 98–104.

-. "Anwendung der Differenzierentzerrung im Fernsehen," rch. elektr. Uebertr., Vol. 9 (1955), pp. 247–251. (3760, 1955).

el, A. van. "Récepteurs de Télévision a Phase Linéaire," *Onde lect.*, Vol. 36 (1956), pp. 48–56.

a) MISCELLANEOUS APPLICATIONS; b) GAMES; c) OPTICS; d) SERVOMECHANISMS

er, F. F. "Information Required for Missile Guidance," *J. Appl. hys.*, Vol. 26 (1955), p. 492. (E76565 (2717, 1955).

no, S. "The Communication Theory Model and Economics," 055 IRE Convention Record, Part 4, pp. 162–181.

akin, F. V. and Karlov, N. V. "Sur la Question de la Sensibilité es Radiomètres," *Zh. Tekh. Fiz. S.S.S.R.*, Vol. 25 (1955), pp. 30–435, 733–741. (E77464, E77465).

eman, J. J. "On Power Spectra and the Minimum Detectable ignal in Measurement Systems," *J. Appl. Phys.*, Vol. 26 (1955), p. 236–240. (D1901, 1955).

dstein, S. J. "A Comparison of Two Radiometer Circuits," roc. IRE, Vol. 43 (1955), pp. 1663–1666.

od, W. C. and Koopmans, T. C., (ed.). *Studies in Econometric ethod. Mon.* 14, New York: Wiley, 324 pp., 1953.

bbs, P. "Le Calcul d'une Limite Supérieure du Module de Erreur Commise sur la Mésure d'une Grandeur Electrique," ev. E., Belg., Vol. 1, No. 8 (1955), pp. 161–172. (E77398).

in, L. R. "Economic Fluctuations in the United States, 1921– 941," *Cowles Comm. Mon.*, No. 11. New York: Wiley, 174 pp., 950.

oopmans, T. C. (ed.). "Activity Analysis of Production and Allo- ation," *Cowles Comm. Mon.*, No. 13. New York: Wiley, 404 pp., 951.

-. "Statistical Inference in Dynamic Economic Models," *Cowles omm. Mon.*, No. 10. New York: Wiley, 438 pp., 1950.

ke, A. S., *et al. Guidance, Coll. Principles of Guided Missiles esign*, New York: Van Nostrand, 729 pp., 1955. (E—L 3761).

Closkey, J. F. and Trefethen, F. N. *Operations Research for anagement*, Baltimore: Johns Hopkins Press, 409 pp., 1954. Rev. roc. IRE, Vol. 43 (July, 1955), pp. 897–898, *Science*, Vol. 122 955), pp. 695–696.

ers, C. N. "Information Retrieval on Structured Content," nform. Theory, Proc. London Symp. (1955), pp. 121–134.

nsted, P. S. "Quality Control and Operations Research," ndustrial Quality Control, Vol. 12 (1955), pp. 3–4, 11–14.

elling, H. von. "Statistische Modelle als Hilfsmittel der Natur- eschreibung," *Mitt. Bl. Math. Stat.*, Vol. 7 (1955), p. 173.

st, J. L. van. "Signaal-ruis verhouding en Informatie," *Ned. . v. Natuurkunde*, Vol. 22 (1956), pp. 233–237.

tner, G. "The Variate Difference Method," *Cowles Comm. Mon.*, o. 5. Evanston, Ill.: Principia Press, 175 pp.

T., Oosterkamp, W. J., and Proper, J. "Limits of Detail Per- eptibility in Radiology Particularly When Using the Image tensifier," *Philips Research Rep.*, Vol. 10 (1955), pp. 141–157.

——, ——. "The Application of the X-Ray Image Intensifier. II The Perception of Small-Object Detail," *Philips Tech. Rev.*, Vol. 17 (1955), pp. 71–77.

Tustin, A. and Boonton, R. C. "The Role of Mathematics in Eco- nomic Prediction and Stabilization," *Int. Soc. Sci. Bull.*, Vol. VI, No. 4, p. 622, 1954. Unesco, Paris.

b)

Newell, A. "The Chess Machine—An Example of Dealing with a Complex Task by Adaption," *Proc. 1955 Western Joint Computer Conf.*, Los Angeles, Calif., 1955, pp. 101–108.

Shannon, E. C. and McCarthy, J. *Automata Studies*, Princeton Univ. Press, 285 pp., 1956. Rev. E. J. Cogan, *Science*, Vol. 124 (1956), p. 493.

c)

Blanc Lapierre, A. and Dumontet, P. "La Notion de Cohérence en- Optique," *Rev. d'Optique*, Vol. 34 (1955), pp. 1–21.

——, Perrot, M., and Péri, G. "Comparaison entre la Transmission de l'Information en Optique et en Radioélectricité," *Optica Acta*, Vol. 2 (1955), pp. 1–5.

Clark Jones, R. "New Methods of Describing and Measuring the Granularity of Photographic Materials," *J. Opt. Soc. Amer.*, Vol. 45 (1955), pp. 799–808.

Croce, P. and Marquet, M. "Determination Optique du 'Bruit de fond' Photographique," *Optica Acta*, Vol. 2 (1955), pp. 107–108.

Dumontet, P. "Sur la Correspondance Objet-image en Optique," *Optica Acta*, Vol. 2 (1955), pp. 53–63.

——. "Correspondance en Amplitude entre un objet Plan et son Image à travers un Système Optique centré," *C. R. Acad. Sci. Paris*, Vol. 242 (1956), pp. 2320–2322.

Gabor, D. "Optical Transmission," *Inform. Theory, Proc. London Symp.* (1955), pp. 26–34.

Kopal, A. (ed.). *Modern Astronomical Optics (Manchester Symposium 1955)*. Amsterdam North Holland Publ. Co., 428 pp. 1955. Rev. *Nature*, Vol. 178 (1956), p. 823.
    "I Information Theory and Optics."
    D. Gabor, *Light and Information*, pp. 17–30.
    A. Blanc Lapierre, M. Picinbono, and M. Savelli, *Mesures sur le Bruit de Fond en Optique*, pp. 31–37.
    E. H. Linfoot, *Noise, Aberrations and the Information Content of Optical Images*, pp. 38–49.
    P. M. Duffieux, *Aspects du Problème de l'Objet*, pp. 50–58.
    D. Gabor, *Collecting Information on Partially Known Objects*, pp. 59–70.
    R. C. Spencer, *Antennas for Radio Astronomy*, pp. 30–162.

Linfoot, E. H. "Resolving Power and Information," *J. Opt. Soc. Amer.*, Vol. 46 (1956), p. 72.

O'Neill, E. L. "Spatial Filtering in Optics," *IRE Trans.*, Vol. IT-2 (June, 1956), pp. 56–65. (E87863).

Picinbono, B. and Savelli, M. "Mémoires sur le Bruit de Fonds en Optique," *Optica Acta*, Vol. 2 (1955), pp. 106–107.

——. "Modèle Statistique suggeré par la Distribution de Grains d'Argent dans les films Photographiques," *C. R. Acad. Sci. Paris*, Vol. 240 (1955), pp. 2206–2208.

——. "Fluctuations de la Transparence Locale d'un Film Photo- graphique Uniformement Impressionné," *C. R. Acad. Sci. Paris*, Vol. 260 (1955), pp. 2296–2298.

Problems in Contemporary Optics, Florence: Inst. Nazionale de Ottica, 700 pp., September, 1954. Rev. in *Optica Acta*, Vol. 3 (1956), p. 152.
    "Optique et Theory de l'Information."
    A. Blanc Lapierre, *Utilisation des Méthodes de l'Analyse Har- monique et de la Théorie de l'Information pour l'Etude de la Cor- respondance Objet-Image en Optique.*
    A. Blanc Lapierre, M. Perrot and G. Péri, *Deux Expériences de Principe suggérant certain Méthodes pour l'Amélioration de la Limite de Résolution d'un Instrument d'Optique.*
    J. Gaultier du Marache, *La Photographie, filtre de fréquences.*
    E. H. Linfoot, *Informational Criteria of Image Quality and Optical Design.*
    G. Toraldo de Francia and M. T. Zoli, *Optical Images Considered as Stochastic Sources.*
    G. Toraldo di Francia, *Capacity of an Optical Channel in the Presence of Noise.*
    A. Maréchal and P. Croce, *Amélioration de la Perception des*

Details des Images par Filtrage Optique des Fréquences Spatiales.

     P. M. Duffieux, *La Limite du Domaine Optique de l'Extension.*

     P. M. Duffieux, *Quantitéi et Qualité d'Information.*

     P. Fellgett, *A Proposal for a Radial-Velocity Rhotometer.*

     E. Ingelstam, *Different Forms of Optical Information and Some Interrelations Between Them.*

Ratcliffe, J. A. "Some Aspects of Diffraction Theory and Their Application to the Ionosphere," *Rep. Prog. Phys.*, Vol. 19 (1956), pp. 188–267.

Toraldo di Francia, G. "Directivity, Super Gain and Information," IRE Trans., Vol. AP-4 (July, 1956), pp. 473–478.

——. "On Resolving Power and Information," *J. Opt. Soc. Amer.*, Vol. 45 (1955), pp. 497–501, Vol. 46 (1956), p. 72. (A6882, 1955) (E77095).

——. "Capacity of an Optical Channel in the Presence of Noise," *Optica Acta*, Vol. 2 (1955), pp. 5–8. (A6875, 1955).

d)

Baker, W. R. G. "Automation," 1955 IRE Convention Record, Part 4, pp. 54–57.

Barker, R. H. "A Signal System for Digital Data Transmission," *Proc. IEE*, Paper No. 1096 M. (D4974, 1955).

Barrett, J. F. and Coales, J. F. "An Introduction to the Analysis of Nonlinear Control Systems with Random Inputs," *Proc. IEE*, C103 (1956), pp. 190–199.

Bibbero, R. J. "Automation, A Survey," *Elect. Eng.*, Vol. 74 (1955), pp. 775–780. (E78778).

Bolz, R. W. "Automation," 1955 IRE Convention Record, Part 4, pp. 58–59.

Booton, R. C. Jr. "The Analysis of Nonlinear Control Systems with Random Inputs," IRE Trans., Vol. CT-1 (December, 1954) pp. 32–34. *Proc. Symp. Nonlinear Circuit Analysis*, Brooklyn, pp. 369–391, 1953.

Bruns, R. A. and Saunders, R. M. *Analysis of Feedback Control Systems. Servomechanisms and Automatic Regulators.* New York: McGraw-Hill, 383 pp., 1955. (D3486, 1955).

Burford, T. M., Rideout, V. C., and Sather, D. S. "The Use of Correlation Techniques in the Study of Servomechanisms," *J. Brit. IRE*, Vol. 15, (1955) pp. 249–258, 526. (E76934).

Carniol, B. and Dykast, K. "Choice of the Bandwidth in Servo-systems Based on the Minimum Dynamic and Noise Errors," *Slaboproudy Obzor*, Vol. 16 (1955), pp. 183-188. (In Czech.)

Casserly, G. and Truxal, J. G. "Measurement and Stabilization of Nonlinear Feedback Systems," Proc. IRE. Abstr., Vol. 44 (March, 1956), p. 389.

Chestnut, H. and Mayer, R. W. *Servomechanisms and Regulating System Design.* New York: Wiley, Vol. II, 384 pp., 1955 and Vol. I, 505 pp., 1951. (D4542, 1955).

Doersam, C. H. "The 'Reasonableness' Check in Automation," 1956 IRE Convention Record, Part 4, pp. 67–77.

Garcin, F. M. J. "La Cybernétique pratique á l'échelle française," *Hommes Tech.* (1955), pp. 818–823. (E74193).

Gardon, R. L. "A Nonlinear Noise Suppression Network for Feedback Control," 1956 IRE Convention Record, Part 4, pp. 46–51.

Hrones, J. A. "A Survey of Some Recent Developments in the Automatic Control Field," *Appl. Mech. Rev.*, Vol. 8 (1955), pp. 501–504.

Laning, J. H. Jr. and Battin, R. H. "Random Processes and Automatic Control," Rev. Proc. IRE, Vol. 44 (October, 1956), p. 1487.

Loeb, J. "Les Servomécanismes à programme," *Automatisme*, Vol. 7 (1956), pp. 235–242. (E86680).

Mathews, M. V. "A Method for Evaluating Nonlinear Servomechanisms," *Trans. AIEE, Appl. Ind.*, (1956), pp. 114–123.

Nejman, M. S. "Sur la Théorie Générale des Signaux et la Théorie Générale du Système Asservis," *Radiotehnike S.S.S.R.*, Vol. 10 (1955), pp. 13–16. (E78656).

O'Neill, S. J. "A Servomechanism Approach to the Problem of Communication for Aircraft Control," IRE Trans., Vol. CS-4, (May, 1956), pp. 129–143.

Petrov, B. N., Popov, E. P., and Voronov, A. A. "Development of the Theory of Automatic Control in the U.S.S.R." Trans. Second All-Union Congress on Theory of Automatic Control, Vol. 1, pp. 13–50. *Izdat. Akad. Nauk. S.S.S.R.*, Moscow, Leningrad, 1955. (In Russian.) (C851, 1956).

Popov, E. P. "On the Use of Harmonic Linearization in the Auto-

matic Control Theory," *C. R. Acad. Sci. U.R.S.S.*, Vol. 1 (1956), pp. 211–214.

Rose, N. J. "Optimum Switching Criteria for Discontinuous Automatic Controls," 1956 IRE Convention Record, Part 4, p. 61–66.

Rosenbrock, H. H. "An Approximate Method for Finding the 'Best Linear Servomechanism,'" *Proc. IEE*, C103, (1956) pp. 260–266. (E88709).

——. "The Integral-of-Error-Squared-Criterion for Servomechanisms," *Proc. IEE*, B102 (1955), pp. 602–608.

Rubinoff, M. "Digital Computers for Real-Time Simulation," *J. Assoc. Comp. Mach.*, Vol. 2 (1955), pp. 186–204.

Schönfeld, —. "Die Wahl der Bandbreite bei Regelstrecken in bezug auf kleinste, sowohl dynamisch als auch durch Rauschen verursachte Regelabweichung," Abstr. *Nachr. Tech.*, Vol. 6 (1956) p. 165.

Sklansky, J. and Ragazzini, J. R. "Analysis of Errors in Sampled Data Feedback Systems," *Trans. AIEE, Appl. Ind.*, No. 18 (1955), pp. 65–71, 74.

——. *Network Compensation of Error-Sampled Feedback Control Systems.* New York: Columbia University, 111 pp., April, 195 (PB118409).

Truxal, J. G. "Network Theory in the Practical Design of Control Systems," Proc. IRE, Vol. 44 (March, 1956), p. 402.

Turner, H. D. "The Automatic Factory," *Penguin Sci. News*, Vol. 37 (1955), pp. 90–110.

Tuttle, K. B. "Design of a Nonlinear Control System with Random Disturbances," *Proc. Natl. Simulation Conf.*, Dallas, 1956.

U.A.W.-C.I.O. "Comm. on Automation. "Automation," *Comm. Aut.*, Vol. 4, pp. 14 -20.

West, J. C. and Nikiforuk, P. N. "The Frequency-Response of a Servomechanism Designed for Optimum Transient Response," *Trans. AIEE, Appl. Ind.*, No. 26 (1956), pp. 234–239.

X. MATHEMATICS: a) STATISTICS;   b) GAMES (THEORY) c) RELAY-ALGEBRA, RELIABILITY;   d) NOISE ANALYSIS

a)

Bartlett, M. S. *An Introduction to Stochastic Processes with Special Reference to Methods and Applications*, Cambridge: Cambridge Univ. Press, 312 pp., 1955. (C939, 1955). (3682, 1955).

Bennett, W. R., Curtis, H. E., and Rice, S. O. "Interchannel Interference in FM and PM Systems Under Noise Loading Conditions," *Bell. Sys. Tech. J.*, Vol. 34 (1955), pp. 601–636. (3089, 1955). (E76580).

Cane, V. R. "Some Statistical Problems in Experimental Psychology," *J. Roy. Stat. Soc.*, Vol. 18B (1956), pp. 177–201.

Favreau, R. R., Low, H., and Pfeffer, I. "Evaluation of Complex Statistical Functions by an Analog Computer," 1956 IRE Convention Record, Part 4, pp. 31–37.

Leeuw, K. de, Moore, E. F., Shannon, E. C., and Shapiro, N. *Computatility by Probabilistic Machines. Automata Studies*, Princeton: Princeton Univ. Press, pp. 183–212, 1956.

Longuet-Higgins, M. S. "Statistical Properties of a Moving Waveform," *Proc. Camb. Phil. Soc.*, Vol. 52 (1956), pp. 234–245.

Sakaguchi, M. "Notes on Statistical Applications of Information Theory," *Rep. Stat. App. Res. Un. Japan. Sci. Eng.*, Vol. (1955), pp. 57–68. (C758, 1956).

Spencer-Brown, G. "Chance and Control (Some Implications of Randomization)," *Inform. Theory, London Symp.* (1955), pp. 8–17.

Ville, J. A. "Leçons sur quelques aspects nouveaux de la Théorie des Probabilités," *Ann. Inst. H. Poincaré*, Vol. 14 (1954), pp. 61–143. (C838, 1955). (E77769).

b)

Farquharson, R. "Strategic Information in Games and in Voting," *Inform. Theory, London Symp.* (1955), pp. 47–54.

c)

Benner, A. H. and Meredith, B. "Designing Reliability into Electronic Circuits," *Natl. Electron. Conf.*, Vol. 10 (1954), pp. 137–145.

Creveling, C. J. "Increasing the Reliability of Electronic Equipment by the Use of Redundant Circuits," Proc. IRE, Vol. 4 (April, 1956), pp. 509–514.

Di Toro, M. J. "Reliability Criterion for Constrained Systems," IRE Trans., PGRQC-8 (September, 1956), pp. 1–7.

fman, D. A. "The Synthesis of Linear Sequential Coding
etworks," *Inform. Theory, London Symp.* (1955), pp. 77–95.

--. *The Synthesis of Sequential Switching Circuits*, M.I.T. Res.
ab. Electronics Tech. Rep. 274, 1954.

--. *A Study of the Memory Requirements of Sequential Switching
ircuits*. Res. Lab. Electronics Tech. Rep. 293, 28 pp. (C1219,
956).

ley, R. S. "Digital Computational Methods in Symbolic Logic
ith Examples in Biochemistry," *Proc. Natl. Acad. Sci.*, Vol. 41
1955), pp. 498–511.

bbert, W. F. "Achieving Operational Effectiveness and Re-
ability with Unreliable Components and Equipment," 1956
RE Convention Record, Part 6, pp. 41–49.

ron, M. E. "Logic, Discovery, and the Foundations of Com-
uting Machinery," IRE Trans., Vol. EC-3 (1954), pp. 2–7.

aly, G. H. "A Method for Synthesizing Sequential Circuits,"
ell. Sys. Tech. J., Vol. 34 (1955), pp. 1045–1079.

ltzer, S. A. "Designing for Reliability," IRE Trans., PGRQC-8
September, 1956), pp. 36–44.

ore, E. F. and Shannon, C. E. "Reliable Circuits Using Less
Reliable Relays," *J. Franklin Inst.*, Vol. 262 (1956), pp. 191–208,
I, pp. 262, 281–297.

skowitz, F. and McLean, J. B. "Some Reliability Aspects of
ystems Design," IRE Trans., PGRQC-8 (September, 1956),
p. 7–36.

eller, R. *On the Synthesis of a Minimal Representation of a Logic
unction*. Air Force Cambridge Res. Center TR-55-104, 13 pp.,
955.

umann, J. von. "Probabilistic Logics and the Synthesis of Reliable
Organisms from Unreliable Components," *Automata Studies*, pp.
3–98. (C1040, 1956).

z, F. M. "Elektrische Netzwerke mit Schaltkontakte," *Fern-
meldetech. Z.*, Vol. 8 (1955), pp. 328–334. (E77140).

sch, J. "Die Matrix in der Schaltungsalgebra zur Planung
elaisgesteuerter Netzwerke," *Archiv. el. Ubertr.*, Vol. 9 (1955),
p. 460–468.

Sarma and Anjan Kumar. "An Introduction to Algebra of
Contractor Relay Circuits," *J. Assoc. Appl. Phys.*, Calcutta
University, Vol. 1 (1955), pp. 72–83. (C110, 1056).

estakov, V. I. "Algebraic Methods of Synthesis of Multicontact
Relay Systems," *C. R. Acad. Sci. U.R.S.S.*, Vol. 99 (1954),
p. 987–990. (1269, 1955).

oboda, A. "The Synthesis of Relay Networks," *Stroje na Zpra-
covani Informaci*, Vol. 2 (1954), pp. 157–208. (In Czech.) (C111,
956).

oboda, F. "The Use of Indeterminate Twovalued Boolean
unction for the Synthesis of Switching Circuits," *Stroje na
Zpracovani Informaci*, Vol. 2 (1954), pp. 209–244. (In Czech.)

iss, G. H. and Kleinerman, M. H. "On the Reliability of Net-
works," *Natl. Electron. Conf.*, Vol. 10 (1954), pp. 128–136.

itzsch, F. "Traditionelle Aussagenlogik und Elektronische
Rechen- und Schaltanlagen," *Elektr. Rdschau*, Vol. 10 (1956),
p. 331–334.

manek, H. "Schaltalgebra," *Radiotechnik*, Vol. 31 (1955), pp.
01–210. (E76592).

nett, W. R. "Methods of Solving Noise Problems," Proc.
RE, Vol. 44 (June, 1956), pp. 609–637.

himovich, V. I. "Sur le Passage du Signal et des Bruits à travers
n Limiteur et un Différentiateur," *Radiotechnika S.S.S.R.*, Vol.
0 (1955), pp. 3–15. (In Russian.)

essin, P. L. "A Bibliography on Noise," IRE Trans., Vol.
T-1 (1955), pp. 15–31.

vis, H. E. and Cooper, G. R. "Direct Reading Probability
Distribution Meter," *Natl. Electron. Conf.*, Vol. 10 (1956), pp.
58–366.

lows, G. E. and Middleton, D. "An Experimental Study of
ntensity Spectra after Half Wave Rectification of Signals in
Noise," *Proc. IEE*, C 103 (1956), pp. 243–248. (E87865).

tkin, L. S. "Interaction of Signal and Noise in an Inertia
Envelop) Detector," *Radiotehnika*, Vol. 11 (1956), pp. 43–53.
In Russian.) (D4283, 1956).

rton, C. W. "On the Use of Gram-Charlier Series to Represent
Noise," *J. Appl. Phys.*, Vol. 27 (1956), pp. 350–356.

dhurst, R. G. and Stewart, J. "The Power Spectrum of a Carrier

Frequency Modulated by Gaussian Noise," Proc. IRE, Vol.
43 (1955), pp. 752–753.

Norton, K. A., Vogler, L. E., Mansfield, W. V., and Short, P. J.
"The Probability Distribution of a Constant Vector plus a Ray-
leigh-Distributed Vector," Proc. IRE, Vol. 43 (October, 1955),
pp. 1354–1361.

Richardson, J. M. "Noise in Driven Systems," IRE Trans., Vol.
IT-1 (March, 1955), pp. 62–65.

Thomson, W. E., "The Response of a Nonlinear System to Random
Noise," *Proc. IEE*, C 102 (1955), pp. 46–48. (C1036, 1955).

Tick, L. J. "Gaussian Outputs from Non-Gaussian Inputs," *J.
Aeron. Sci.*, Vol. 22 (1955), pp. 3, 208.

Wooding, R. A. "An Approximate Joint Probability Distribution
for Wave Amplitude and Frequency in Random Noise," *New
Zealand J. Sci. Tech.* B 36 (1955), pp. 537–544. (3401, 1955).
Nature, Vol. 176 (1955), pp. 564–565. (E77868).

XI. PULSE MODULATION, MULTIPLEX, CODING

Bàlser, M. and Silverman, R. A. "Coding for Constant-Data-Rate
Systems; 2. Multiple Error Correcting Codes," Proc. IRE,
Vol. 43 (June, 1955), pp. 728–733. (2741, 1955).

Benz, F. "Die Impulskodemodulation und ihre weitere Entwick-
lungs-Möglichkeiten," *El Masch.*, Vol. 76 (1956), pp. 73–78.

--- ·. "Masznahmen zur Störverminderung und Erhöhung der
Kanalkapazität bei Impulskodemodulation," *Arch. el. Ubertr.*,
Vol. 9 (1955), pp. 299–306, 381–387. (E77848, 77852).

Bosse, G. "Codemodulation für Trägerfrequenztechnik," *Nachr.
Tech. Fachber.*, Vol. 6 (1956), pp. 16–18.

Brock, R. L. and McCarty, R. C. "On the Modulation Levels in a
Frequency Multiplexed Communication System by Statistical
Methods," IRE Trans., Vol. IT-1 (March, 1955), pp. 54–59.
Abstr. Proc. IRE, Vol. 43 (July, 1955), p. 893.

Burks, A. W., McNaughton, R., Pollmar, C. H., Warren, D. W.,
and Wright, J. B. "The Folded Tree," *J. Franklin Inst.*, Vol.
260 (1955), pp. 9–24.

---. "Complete Decoding Nets: General Theory and Minimality,"
J. Soc. Ind. Appl. Math., Vol. 2 (1954), pp. 201–243.

Carassa, F. "Pulse Multiplex Telephony System with High Precision
and Phase Stability," *Alta Frequenza*, Vol. 24 (1955), pp. 139–177.
(3403, 1955).

Cohen, J. W. and Jung, M. W. "Some Calculations on a Method for
Signal Testing," *Inform. Theory, Proc. London Symp.*, Abstr.
(1955), p. 400.

Colander, R. C. and Kortman, C. M. "A Transistorized FM/FM
telemetering System," IRE Trans., Vol. TRC-1 (June, 1955),
p. 20–24.

Corsepius, M., Logemann, H., and Vogt, K. "Untersuchungen
über ein Acht-Kanal-Telegraphiesystem mit automatischer
Fehlerkorrektur," *Nachr. Tech. Z.*, Vol. 9 (1956), pp. 306–309.
(E87023).

--- and Vogt, K. "Ueber Beobachtungen und Erfahrungen mit der
TOM Anlage auf Kurzwellenübertragungswegen beim Funkamt
Frankfurt am Main," *Nachr. Tech. Z.*, Vol. 9 (1956), pp. 55–59.

Das, J. "Quantitative Noise Reduction in Pulse Time Modulation,"
Electronic Eng., Vol. 27 (1955), pp. 406–409. (E77875).

---. "Pulse-Slope Modulation—a New Method of Modulating
Video Pulses and its Possible Application on Line Circuits,"
Indian J. Phys., Vol. 28 (1954), pp. 449–462. (2737, 1955).

---. "Signal Analysis and Audiocharacteristics of Pulse Slope
Modulation," *Electronic Eng.*, Vol. 27 (1955), pp. 482–487.
(561, 1956).

Diamond, J. M. "Checking Codes for Digital Computers," Proc.
IRE, Vol. 43 (1955), pp. 487–488.

Ditl, A. "Practical Application of the Fundamental Theorem of
Pulse Modulation (Theory)," *Slaboproudy Obzor*, Vol. 15 (1954),
pp. 476–482. (In Czech.)

Ducot, C. "Procédé Technique pour l'Amélioration des Perform-
ances des Faisceaux Hertziens en Téléphonie," *Ondé Elect.*, Vol.
35 (1955), pp. 41–54.

Elias, P. "Predictive Coding," IRE Trans., Vol. IT-1 (March,
1955), pp. 16–24, 24–33. Abstr. Proc. IRE, Vol. 43 (July, 1955),
p. 893.

---. "Coding for Noisy Channels," 1955 IRE Convention Record,
Part 4, pp. 37–46.

---. "Coding for Two Noisy Channels," *Inform. Theory, London
Symp.* (1955), pp. 61–77.

——. *Error Free Coding*, M.I.T. Res. Lab. Electronics Tech. Rep., p. 285, 1955.

Evans, W. E. and Lowe, R. F. "Quadriphase: a New Approach to Time Division Multiplexing," *Electrical Eng.*, Vol. 74 (1955), pp. 685–688.

Fagot, J. "Problèmes Physiques posés par les Transmission sur Faisceaux Hertziens," *Ondé Elect.*, Vol. 36 (1956), pp. 7–22.

Filipowsky, R. "Ein Mehrkanal Impulsübertragungssystem mit automatisch veränderlicher Impulszahl," *Ost. Z. Teleph. Telegr. Fernsehtech.*, Vol. 9 (1955), pp. 113–129, 147–158.

——. "Electrical Pulse Communication Systems," *J. Brit. IRE*, Vol. 15 (1955), pp. 451–467, 483–504, Vol. 16 (1956), pp. 40–58. (261, 562, 1956).

Flanagan, J. E. "Coding to Achieve Markov Type Redundancy," *J. Math. Phys.*, Vol. 33 (1954), pp. 258–268.

Golay, M. J. E. "Notes on Hybrid Coding," Proc. IRE, Vol. 43 (May, 1955), p. 625.

Grilse, H. J. "Eine Mehrfach Impuls-Zeitmodulation," *Arch. el. Ubertr.*, Vol. 9 (1955), pp. 571–572. (E82672).

Hahn, C. "Les Equipements á modulation Frequency-shift et leur Emploi dans la Télécommande, la Télésignalisation, et la Télé-protection," Rev. *Brown Boveri*, Vol. 42 (1956), pp. 288–292.

Harris, L. R. F. "Time-Sharing as a Basis for Electronic Telephone Switching," *Proc. IEE*, B 103 (1956), pp. 722–742.

Hermann, P. K. "Optische Impulstechnik," *Imp. t.*, pp. 227–258.

Herreng, P., Blondé, R., and Dureau, G. "Système de Transmission Téléphonique Multiplex à Modulation par Impulsions codées," *Cables et Transm.*, Vol. 9 (1955), pp. 144–160. (3086, 1955).

Hollis, R. M. "Optimum Slicing Level in a Noisy Binary Channel," Proc. IRE, Vol. 44 (November, 1956), pp. 1062–1063.

Holzer, J., Missriegler, G., Niedermayer, E., Putzschi, H., and Zemanek, H. "Ein Versuchsmodell für Impuls-Zahl-Modulation," *Oesterr. Zeits. Telegr. Teleph. Funktechnik*, Vol. 8 (1955), p. 855.

Holzwarth, H. "Probleme der Mehrfachausnutzung von Nachrichtenwegen mit Pulsmodulation," *Imp. t.*, pp. 97–124. (See Winckel XI, p. 46).

Huffman, D. A. "A Linear Circuit Viewpoint on Error-Correcting Codes," IRE Trans., Vol. IT-2 (September, 1956), pp. 20–28. Abstr. Proc. IRE, Vol. 44 (November, 1956), p. 1643.

IRE Standards Comm., "IRE Standards on Pulses: Methods of Measurement of Pulse Quantities," Proc. IRE, Vol. 43 (November, 1955), pp. 1610–1616.

Jordan, D. B., Greenberg, H., Eldredge, E. E., and Serniuk, W. "Multiple Frequency Shift Teletype Systems," Proc. IRE, Vol. 43 (November, 1955), pp. 1647–1655. (564, 1956).

Kaufman, H. and King, E. H. "Spectral Power Density Functions in Pulse Time Modulation," IRE Trans., Vol. IT-1 (March, 1955), pp. 40–46. (D4583, 1955).

Knudtzon, N. *Theoretical Comparison of Fluctuation Noise Characteristics for Various Modulation Methods with Special View to Multi-Channel System Planning*, Thesis Delft, 216 pp., 1957.

Komamiya, Y. "Application of Logical Mathematics to Information Theory," *Proc. Third Japanese Natl. Congr. Appl. Mechanics*, pp. 437–442, 1953; Science Council Japan, Tokyo, 1954. (C1100, 1956).

Kulikowsky, R. and Plebanski, J. "Optimum Characteristics of Linear Pulse Systems," *Bull. Acad. Polon. Sci.*, Cl IV, Vol. 3 (1955), pp. 23–28.

Laemmel, A. E. "A General Class of Codes and Their Physical Realization," *Inform. Theory, Proc. London Symp.* 1955.

Magasiny, I. P. "A Versatile Automatic Data Separation System for Pulse Multiplex Telemetering Systems," IRE Trans., Vol. TRC-1 (March, 1955), pp. 5–15.

Mass, J. "Coding for Remote Control," IRE Trans., Vol. TRC-1 (March, 1955), pp. 1–4.

Mayer, H. F. "Prinzipien der Pulse-Code Modulation," *Verlag R. Oldenbourg*, Munich, 76 S, 1955.

McMillan, B. "History of a Problem," *J. Soc. Ind. Appl. Math.*, Vol. 3 (1955), pp. 119–128.

Medhurst, R. G. "RF Bandwidth of Frequency-Division Multiplex Systems Using Frequency Modulation," Proc. IRE, Vol. 44 (February, 1956), pp. 189–199.

Meyer Eppler, W. *Anwendung der Informationstheorie auf Impulsprobleme Impulstechnik*, Springer, pp. 40–59, 1955.

Oden, H. "Ueber den Gebrauch von Codes in der Fernschreib- und Fernsprechtechnik," *S.E.G. Nachr.*, Vol. 4 (1956), pp. 174–180.

Ott, F. "Automatische Fehler-Detektor und Korrektur Anlage fü Fernschreibverbindungen," *Hasler Mitt.*, Vol. 15 (1956), pp. 1–7 (E88084). *Tech. Mitt. P.T.T.*, Vol. 34 (1956), pp. 223–229 376–381.

Radius, K. "Mehrfachübertragung durch Amplitudenvielfach," *Nachr. Tech. Z.*, Vol. 9 (1956), pp. 403–407.

Reichardt, W. "Binarschritte und gerichtete Impulse im Licht der Informationstheorie," *Hochfrq. Elektroak.*, Vol. 64 (1956) pp. 76–83.

Sanchez, M. and Popert, F. "Ueber die Berechnung der Spektren modulierter Impulsfolgen," *Archiv. el. Ubertr.*, Vol. 9 (1955) pp. 441–452.

Schauffler, R. "Ueber die Bildung von Codewörtern," *Arch. Elektr Ubertr.*, Vol. 10 (1956), pp. 303–314. (E87748).

Schroeter, F. "Modellvorschlaege für Quantisierung und Speiche rung," *Nachr. Tech. Fachber.*, Vol. 3 (1956), pp. 72–81.

Schuetzenberger, M. P. "Une Théorie Algebrique du Codage," *C. R. Acad. Sci. Paris*, Vol. 242 (1956), pp. 862–864.

——. "On an Application of Semi-Group Methods to Some Problem in Coding," IRE Trans., Vol. IT-2 (September, 1956), pp 47–60, Abstr. Proc. IRE, Vol. 44 (November, 1956), p. 1643.

Shea, J. E. and Ordnung, P. F. "Feedback-Controlled Length Modulated Pulse Generator," Proc. IRE. Abstr., Vol. 44 (March 1956), p. 389.

Siforov, V. I. "Theory of Ideal Coding for Binary Transmission," *Radio Technics and Electronics*, Moscow, Vol. 1 (1956), pp 407–417.

Siforov, V. S. "Error Reduction of Systems with Correcting Codes," *Radio Technics and Electronics*, Moscow, Vol. 1 (1956), pp 131–142.

Silverman, R. A. "On Binary Channels and Their Cascades," IRE Trans., Vol. IT-1 (September, 1955), pp. 19–27. Quart. Prog Rep. M.I.T., Res. Lab. Electronics (July, 1955), pp. 61–63.

Six, W. "The T.O.R. (Teletype on Radio) Circuits for the Argentin Radio Links," *Tijdschr. Ned. Radio Gen.*, Vol. 20 (1955), pp 137–159. (3745, 1956).

Slepian, D. "A Class of Binary Signalling Alfabets," *Bell. Sys Tech. J.*, Vol. 35 (1956), pp. 203–234. Abstr. *Inform. Theory Proc. London Symp.*, 1955.

——. "A Note on Two Binary Signaling Alphabets," IRE Trans. Vol. IT-1 (June, 1956), pp. 84–86. (E87860).

Slow, J. L. "Intermodulation Noise in V. H. F. Multichanne Telephone Systems," *J. Brit. IRE*, Vol. 15 (1955), pp. 67–83.

Smith, J. R. W. and Slow, J. L. "Energy Distribution in a Wav Frequency Modulated by a Multichannel Telephone Signal," *A.T.E.J.*, Vol. 12 (1956), pp. 182–202. (E87870).

Staras, H. "The Statistics of Combiner Diversity," Proc. IRE Vol. 44 (August, 1956), pp. 1057–1058.

Steffenhagen, K. "Geräuschverminderung bei der Impuls Cod Modulation," *Nachr. Tech.*, Vol. 5 (1955), pp. 243–248. (E77127)

Stone, R. P., Mueller, C. W., and Webster, W. M. "A Time Sampling and Amplitude Quantizing Tube," Proc. IRE, Vo 43 (August, 1955), pp. 960–966. (3458, 1955).

Tickhonov, V. I. "Action des Impulsions Rectangulaires a Duré età Intervalle à Répartition Aléatorire sur un Détecteur Linéaire," *Radiotehnika S.S.S.R.*, Vol. 10 (1955), pp. 11–15. (In Russian.)

Tomovich, R. "Circuits with Quantized Feedback," IRE Trans. Vol. CT-2 (June, 1955), pp. 209–213. Abstr. Proc. IRE, Vo 43 (September, 1955), p. 1154.

Villepelet, J. "Etude Théorique et Expérimentale du Brouillag mutuel entre Systèmes de Radiocommunications," *Ann. Télécomm* Vol. 10 (1955), pp. 264–276.

Watanabe, S. "Binary Coding and Isometric Transformation in a Finite Metric Boolean Algebra," *Rep. Unio. Electrio. Commun* Vol. 7 (1955), pp. 17–40. (C1178, 1956).

Watson, R. A. and Hudson, O. K. "Transmitting System Usin Delta-Modulation," *Electronics*, Vol. 29 (1956), pp. 164–166.

Winckel, F. (ed.). *Impulsetechnik (Vortragsreihe)*, Berlin: Springe 346 pp., 1956.

Yates-Fish, N. L. and Fitch, E. "Signal Noise Ratios in Pulse Cod Modulation," *Proc. IEE*, B 102 (1955), pp. 204–210. (178 1955). (D1701, 1955).

Zaremba, S. K. "Spacing Problems in Abelian Groups," *Proc. In Congr. Math.*, Amsterdam (1954), p. 74.

Zetterberg, L. H. "A Comparison between Delta and Pulse Cod Modulation," *Ericson Tech.*, Vol. 11 (1955), pp. 95–154. (E76567 *Inform. Theory, Proc. London Symp.* 1955.

orrespondence

ermination of Redundancies in a ...of Patterns*

the above article,[1] Glovazky states
the minimum number of cells that
be scanned to uniquely identify his
arbitrary patterns is four, namely, 1,
8.

would like to submit that the patterns
be uniquely identified by scanning
r cells 3, 4, and 9 or 3, 5, and 9—in
r case one cell less than his "minimum
issible." (See Fig. 1.)

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eceived by the PGIT, March 19, 1957.
. Glovazky, IRE TRANS., vol. IT-2, pp. 151–
December, 1956.

	3	4	9
1	1	0	0
2	0	0	1
3	0	1	0
4	0	1	1
5	1	1	0
6	1	1	1

or ⟵ ⟶ No two alike

	3	5	9
1	1	1	0
2	0	1	1
3	0	1	0
4	0	0	1
5	1	0	0
6	1	0	1

Fig. 1.

hor's Comment[2]

iekeman is quite correct in his
rvation that the reduced schedule
ented in the article does not represent
absolute minimum number of cells.
was meant to be conveyed by the last
of the paper, the number of cells
h appears in the reduced schedule
nds on the particular "scanning path,"
on the particular sequence of columns
en. The significance of the proposed
od is that, regardless of the sequence,
can always be sure that the resulting
dule will not contain more than P-1
This by itself constitutes a great
ovement in many practical cases
e the number of cells greatly exceeds
number of patterns ($C \gg P$).
he problem of finding the absolute
mum schedule (or schedules) is by
r of magnitudes more difficult, because
e is no *a priori* knowledge of what
mn sequence will yield the most
ent reduction. The optimal sequence
be approached if one places at the
nning of the schedule those columns in
h the ratio between "ones" and
s" is closest to unity (*i.e.*, those
nns which exhibit the highest degree
ncertainty). This method, however,
when the ones-to-zeros ratio in all
given columns is about the same. Any
ral method of finding the absolute
mum schedule involves, to the author's
rledge, examination of all possible
mn sequences -a task which in most
is too laborious to be of any practical
In such cases the simplicity and
pactness of the separation scheme
ly outweighs the limitations mentioned
e.

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eceived by the PGIT, April 13, 1957.

Letter from Dr. McCluskey[3]

In connection with the article by
Glovazky,[1] I would like to point out a
method for determining the redundancies
in a set of patterns without a scanning
path being assumed. This method should
lead to a specification of the fewest cells
which must be scanned or to all alternative
sets of cells which can be scanned to
identify the patterns.

I will assume that the set of patterns is
specified by a code matrix, Fig. 2, (this is
the transpose of Glovazky's code schedule).
Each row of this matrix represents a cell
and each column a pattern. If the ith cell
of the jth pattern is black, the entry in the
ith row and jth column of the matrix
(a_{ij}) is 1, otherwise it is 0. From this
matrix another matrix called the pair
matrix, Fig. 3, is to be formed in which each
column is derived by taking the sum
modulo two of (see Fig. 4) the elements
from a pair of columns of the code matrix.
This is to be done for all columns of the
code matrix so that if the code matrix has
r rows and m columns, the pair matrix
will have r rows and $mC2$ columns. The
column of the pair matrix derived from
columns j and k of the code matrix will
have a 1 in only those rows which corre-
spond to cells in which the jth and kth
patterns differ. In order to distinguish
between the jth and kth patterns, at least
one cell for which the corresponding row
has a 1 in the jk column of the pair matrix
must be scanned. In order to distinguish
among all pairs of patterns, a set of cells
such that the corresponding rows have at
least one 1 in each column of the pair
matrix must be scanned. Thus the sets of
cells to be scanned can be determined by
discovering each set of rows which has the
property that every column has a 1 in at
least one row of the set.

³Received by the PGIT, March 26, 1957.

There are several techniques known for
picking such a set of rows. This is exactly
the problem which arises in minimizing
Boolean functions; the pair matrix corre-
sponds directly to a prime implicant
table.[4]

For the pair matrix the following pre-
liminary reduction is possible: if there are
two columns j and k such that column k
has a 1 in all rows in which column j has
a 1, column k can be deleted. In Fig. 3
column 4, 5 has 1's in every row in which
column 5, 6 has 1's, therefore column 4, 5
can be deleted. Satisfying the require-
ments for column j will automatically
satisfy those for column k. Application of
this method to the set of patterns given in
Glovazky's paper gives two scanning
sequences containing only three cells:
(3, 4, 9) and (3, 5, 9).

While this technique is longer than that
using the code schedule and reduced code
schedule,[1] it considers all possible scanning
paths simultaneously and therefore is
roughly equivalent to $n!$ applications of
the code schedule technique (where n
equals the number of cells). It should be
useful where all scanning paths are to be
considered.

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⁴This was first introduced by W. V. Quine, "The
problem of simplifying truth functions," *Amer. Math.
Monthly*, vol. 59, pp. 521–531; October, 1952.
Techniques for selecting the sets of rows are given
by E. J. McCluskey, Jr., "Minimization of Boolean
functions," *Bell Sys. Tech. J.*, vol. 35, pp. 1417–1444;
November, 1956.

Author's Comment[5]

McCluskey's pair matrix presents a
new formulation of the reduction problem,

⁵Received by the PGIT, April 20, 1956.

Fig. 2—Code matrix for code schedule given by Glovazky.[1]

			Patterns				
		1	2	3	4	5	6
	1 A	1	0	0	1	1	1
	2 B	1	1	1	0	1	1
	3 C	1	0	0	0	1	1
	4 D	0	0	1	1	1	1
Cells	5 E	1	1	1	0	0	0
	6 F	0	0	1	0	0	0
	7 G	0	1	0	1	1	1
	8 H	1	1	1	1	0	1
	9 J	0	1	0	1	0	1

Fig. 3—Pair matrix derived from code matrix of Fig. 1.

							Pairs of Patterns								
		1/2	1/3	1/4	1/5	1/6	2/3	2/4	2/5	2/6	3/4	3/5	3/6	4/5	4/6	5/6
	1 A	1	1	0	0	0	0	1	1	1	1	1	1	0	0	0
	2 B	0	0	1	0	0	0	1	0	0	1	0	0	1	1	0
	3 C	1	1	1	0	0	0	0	1	1	0	1	1	1	1	0
	4 D	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0
Cells	5 E	0	0	1	1	1	0	1	1	1	1	1	1	0	0	0
	6 F	0	1	0	0	0	1	0	0	0	1	1	1	0	0	0
	7 G	1	0	1	1	1	1	0	0	0	1	1	1	0	0	0
	8 H	0	0	0	1	0	0	0	1	0	0	1	0	1	0	1
	9 J	1	0	1	0	1	1	0	1	0	1	0	1	1	0	1

Columns that can be deleted

Fig. 4—Sum modulo two.

x	y	sum modulo 2 of x and y
0	0	0
0	1	1
1	0	1
1	1	0

but by no means simplifies it. In the origi
matrix the objective was to delete as ma
rows as possible without destroying
uniqueness of the columns; in the r
matrix the objective is to delete as ma
rows as possible without including a
column which comprises entirely of zer
Either of these objectives calls for a p
cedure which is too long and too invol
to be useful.

The pair matrix contains $P(P - 1$
columns as compared with P columns
the original matrix. The prelimin
deletions alone require in the order
$P^4/8$ operations since they involve
comparison of all possible column p
in the pair matrix. The subsequent
duction process requires even a grea
number of operations, with the result t
the total amount of labor involved
roughly equal, as McCluskey no
to that of repeatedly applying my sepa
tion procedure to all possible colu
sequences.

Thus the pair matrix approach con
tutes one of the general methods to wh
I referred in my rebuttal to Riekema
comments. In cases where the deletion
$C - P + 1$ cells is satisfactory, the sep
tion method is definitely preferable.

ARTHUR GLOVA

Contributors

A. V. Balakrishnan (S '43—A '55—M '56) was born in India on December 4, 1922. He received the Bachelor's and Master's degrees in physics from the University of Madras, India. He came to the United States in 1947 on a two-year Indian Government scholarship. In 1950, he was awarded the Master's degree in electrical engineering and in 1954, the Ph.D. degree in mathematics from the University of Southern California, Los Angeles, Calif.

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While doing graduate work, he was a laboratory associate, teaching assistant, and lecturer at the University of Southern California. He was also an assistant instructor in the Mathematics Department, Yale University, New Haven, Conn.

From 1954 to 1956, he was with RCA, Camden, N.J., working on communication and control problems, including multipath transmission of video signals, noise cancellation systems, and nonlinear filters. He is now an assistant professor at the University of Southern California.

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m February, 1948, to August, 1951, s employed by Melpar, Inc., Alexan- Va. During this time he was involved earch design and development of ppm unications system, voice channel ression as well as other applied infor- n theory projects. From August, 1951 ovember, 1956, he was employed by adiation Laboratory of Johns Hopkins ersity. At the Radiation Laboratory he project supervisor of the group in e of developing automatic airborne analyzer systems. Dr. Blasbalg also ed on statistical detection theory and mathematical theory of observation. 56 he became a research scientist and onsultant.

October, 1956, Dr. Blasbalg joined staff of Electronic Communications, Baltimore, Md., and is presently ved in applied statistical decision y, observation theory and automatic ving systems.

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❖

Deutsch (A '46—M '55) was born w York, N.Y. on September 19, 1918. eceived the B.E.E. degree in 1941 from Cooper Union, and the M.E.E. and D.E.E. degrees in 1947 and 1955 from the Polytechnic In- stitute of Brooklyn.
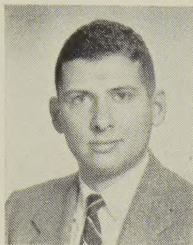
From 1935 to 1940, he was an electric motor technician, and was thereafter employed as a de- signer of electro- mechanical equip- ment until 1944 when

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oined the U.S. Navy. He was an ronics engineer from 1950 to 1954, at olytechnic Research and Development oany and is at present with the owave Research Institute.

Deutsch was a physics instructor at er College during 1943-1944, and was structor in radio and television from to 1950. Since 1951 he has taught at Polytechnic Institute of Brooklyn, e he is now an associate professor of ical engineering. He has also been iated with the Electrical Engineering rtment of the City College of New since 1955.

Deutsch is the author of "Theory Design of Television Receivers," and

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❖

Stephen G. Margolis (S '53—M '56) was born in Philadelphia, Pa. on December 15, 1931. He attended the University of Pennsylvania, where he received the B.S. degree in E.E. in June, 1953. From 1953 to 1955 he was a research assistant in the Research Laboratory of Elec- tronics, M.I.T., Cam- bridge, Mass., work- ing on instrumenta-

S. G. MARGOLIS

tion for analysis of random signals and on the application of correlation techniques to the measurement of transfer functions. After receiving the S.M. degree in E.E. in June, 1955, he joined the Communications Research Group at the Jet Propulsion Laboratory, Cali- fornia Institute of Technology in Pasadena, Calif., where he worked on the design and analysis of phase-locked systems. Since August, 1956 he has been with the Bettis Atomic Power Division, Westinghouse Electric Corporation, Pittsburgh, Pa., where his work concerns control systems for power reactors.

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❖

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During World War II, he was a research associate at the Radio Research Lab- oratory at Harvard, and worked in the field of electronic counter-measures.

D. MIDDLETON

From 1947 to 1949, he was a research fel- low in electronics at the Electronics Research Laboratory of the Division of Applied Science at Harvard. In 1949, he became assistant professor of applied physics in the same

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At present, Dr. Middleton's principal field of research is in statistical communication theory, including applications in electronics, electron physics, information theory, and system design and evaluation. He is also engaged in the study of various problems in applied mathematics which are related to these fields.

He is a fellow of the American Physical Society, a member of the American Math- ematical Society, New York Academy of Sciences, and the Society for Industrial and Applied Mathematics, Phi Beta Kappa, and Sigma Xi. He was a National Research Council predoctoral fellow in physics from 1946 to 1947, and he received the National Electronics Conference Award (with W. H. Huggins), in 1956.

❖

Louis A. Ule was born on March 22, 1916, in Cleveland, Ohio. He received the M.S. degree in mathematics from DePaul Univer- sity, Chicago, Ill. in 1947. From 1946 to 1950 he was an in- structor in the Elec- trical Engineering Department of the American Television Institute, Chicago, and during 1950 was employed as a tele- vision receiver design engineer at Motorola,

L. A. ULE

Inc., Chicago, where he specialized in in- terference reduction and in the design of sweep circuits and associated components. From 1951 to 1957 he was employed as supervisor of the Servo Mechanism and Computer Groups in the Engineering De- partment of Gilfillan Bros. Inc., Los Angeles, Calif. In addition to his work on automatic control systems, he conducted noise studies and actively participated in the design of microwave receivers, and in countermeasure and missile guidance studies. Mr. Ule has continued his education with advanced courses in the U.C.L.A. and University of Southern California graduate schools.

At present he is employed as Assistant Director of Engineering of the Potter Pacific Corporation, Malibu, Calif. In this capacity he is directing varied military programs and is especially active in the development of automatic control systems for industry.
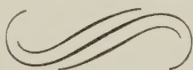
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